EE407: Notable Antennas Equations

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Antennas:

Fields in terms of Magnetic Vector Potential:	$\boldsymbol{B}_s = \nabla \times \boldsymbol{A}_s \qquad \boldsymbol{B} = \mu \mathrm{H}$	$\boldsymbol{E}_s = -j\omega\left[\frac{1}{k^2}\nabla\nabla\cdot\boldsymbol{A}_s - j\omega\boldsymbol{A}_s\right]$	
Dipole centered at 0 A_s	$\boldsymbol{A}_s(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_{V_o}\frac{\boldsymbol{J}_s(\boldsymbol{r}')}{\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}e^{-jk\lvert\boldsymbol{r}-\boldsymbol{r}'\rvert}dv'$	$\boldsymbol{A}_s(\boldsymbol{r}) \approx \frac{\mu}{4\pi}\int_{-\Delta l/2}^{\Delta l/2}\frac{I_o\boldsymbol{a}_z}{(r-z'\cos\theta)}e^{-jk(r-z'\cos\theta)}dz'$	
Infinitesimal Dipole A_s:	$\boldsymbol{A}_s(\boldsymbol{r}) \approx \frac{\mu I_o \Delta l}{4\pi r}e^{-jkr}(\cos\theta\,\boldsymbol{a}_r - \sin\theta\,\boldsymbol{a}_\theta)$		
Infinitesimal Dipole Near Fields:	$\boldsymbol{E}_s = -\frac{j\eta I_o\Delta l}{4\pi r^3 k}\left[2\cos\theta\,\boldsymbol{a}_r + \sin\theta\,\boldsymbol{a}_\theta\right]e^{-jkr}$		
Infinitesimal Dipole Far Fields:	$E_{\theta s} = \frac{j\eta k I_o \Delta l}{4\pi r}\sin\theta\, e^{-jkr}$	$H_{\phi s} = \frac{jkI_o\Delta l}{4\pi r}\sin\theta\, e^{-jkr} = \frac{E_{\theta s}}{\eta}$	
Cartesian – Spherical Conversion:	$\begin{bmatrix}A_r\\A_\theta\\A_\phi\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix}$	$\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}A_r\\A_\theta\\A_\phi\end{bmatrix}$	
	$d\mathbf{S} = r^2\sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$	$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$	
Power Equations:	$\wp_{avg} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{S}\} = \frac{1}{2}\mathrm{Re}\{\boldsymbol{E}_s \times \boldsymbol{H}_s^*\}$	$P_{rad} = \iint_S \wp_{avg}\cdot d\boldsymbol{s}$	$P_{rad} = \int_0^{2\pi}\int_0^{\pi}\left[\wp_{avg}(r=r_o)\cdot\boldsymbol{a}_r\right]r_o^2\sin\theta\, d\theta\, d\phi$
Radiation Resistance:	$P_{rad} = I_{s,rms}^2 R_{rad} = \frac{1}{2}I_{s,peak}^2 R_{rad}$		
Infinitesimal Dipole Radiation Resistance:	$\wp_{avg} = \frac{1}{2}\frac{\lvert E_{\theta s}\rvert^2}{\eta_o}\boldsymbol{a}_r = \frac{1}{2}\eta_o\lvert H_{\phi s}\rvert^2\boldsymbol{a}_r$	$P_{rad} = \frac{(120\pi)4\pi^2 I_o^2\Delta l^2}{12\pi\lambda^2} = 40\pi^2\left(I_o\frac{\Delta l}{\lambda}\right)^2$	
	$P_{rad} = \frac{1}{2}I_o^2 R_{rad} = 40\pi^2\left(I_o\frac{\Delta l}{\lambda}\right)^2$	$R_{rad} = 80\pi^2\left(\frac{\Delta l}{\lambda}\right)^2$	
Dipole Antenna:	$I_s(z') = I_o\sin\left[k\left(\frac{l}{2}-\lvert z'\rvert\right)\right]$	$dE_{\theta s} = \frac{j\eta k I_s(z')dz'}{4\pi r}\sin\theta\, e^{-jk(r-z'\cos\theta)}$	$E_{\theta s} = \int dE_{\theta s}$
	$E_{\theta s} \approx j\frac{\eta I_o e^{-jkr}}{2\pi r}F(\theta)$	$F(\theta) = \frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}$	$H_{\phi s} = \frac{E_{\theta s}}{\eta} \approx j\frac{I_o e^{-jkr}}{2\pi r}F(\theta)$

Half-Wave Dipole in Air:	$F(\theta) = \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$	$E_{\theta s} \approx j\dfrac{\eta_o I_o}{2\pi r} e^{-jkr} \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$	$H_{\phi s} \approx j\dfrac{I_o}{2\pi r} e^{-jkr} \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$	
	$\mathcal{P}_{avg} = \dfrac{1}{2}\eta_o \lvert H_{\phi s}\rvert^2 \boldsymbol{a}_r$	$P_{rad} = \int_0^{2\pi}\int_0^{\pi} \left[\dfrac{\eta_o}{2}\left(\dfrac{I_o}{2\pi r_o}\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2\right] r_o^2 \sin\theta\, d\theta\, d\phi$	$P_{rad} = 36.56\, I_o^2 = \frac{1}{2} I_o^2 R_{rad}$ $R_{rad} = 2(36.56) \approx 73\,\Omega$	
Antenna Impedance:	$S_s = P + jQ$	$P = P_{rad} + P_{loss} = \underbrace{\frac{1}{2}\lvert I_s\rvert^2 R_{rad}}_{\text{Power radiated by the antenna}} + \underbrace{\frac{1}{2}\lvert I_s\rvert^2 R_{loss}}_{\text{Power dissipated by the antenna}}$	$Q = \underbrace{\frac{1}{2}\lvert I_s\rvert^2 X_A}_{\text{Power stored in the near field of the antenna}}$	$\eta_r = \dfrac{P_{rad}}{P_{rad} + P_{loss}} = \dfrac{R_{rad}}{R_{rad} + R_{loss}}$
Monopole Impedance:	$(Z_A)_{monopole} = \frac{1}{2}(Z_A)_{dipole}$	$(Z_A)^{quarter-wave}_{monopole} = \frac{1}{2}(Z_A)^{half-wave}_{dipole} = \frac{1}{2}(73 + j42.5)\Omega = (36.5 + j21.25)\Omega$		
Small Loop Antenna:	$\boldsymbol{A}_s(\boldsymbol{r}) \approx \dfrac{\mu I_o \Delta S}{4\pi r^2}(1 + jkr)\, e^{-jkr} \sin\theta\, \boldsymbol{a}_\phi$	$H_{rs} = jk\dfrac{I_o \Delta S}{2\pi}\cos\theta\left[\dfrac{1}{r^2} - \dfrac{j}{kr^3}\right] e^{-jkr}$	$H_{\theta s} = jk\dfrac{I_o \Delta S}{4\pi}\sin\theta\left[\dfrac{jk}{r} + \dfrac{1}{r^2} - \dfrac{j}{kr^3}\right] e^{-jkr}$	$E_{\phi s} = -j\eta k\dfrac{I_o \Delta S}{4\pi}\sin\theta\left(\dfrac{jk}{r} + \dfrac{1}{r^2}\right) e^{-jkr}$
Small Loop Far Fields:	$H_{\theta s} = -k^2 \dfrac{I_o \Delta S}{4\pi r}\sin\theta\, e^{-jkr}$	$E_{\phi s} = \eta k^2 \dfrac{I_o \Delta S}{4\pi r}\sin\theta\, e^{-jkr}$		
Small Loop Radiated Power and resistance:	$P_{rad} = \dfrac{\eta_o k^4 I_o^2 \Delta S^2}{12\pi}$ replace η_0 with η	In air: $P_{rad} = \dfrac{160\pi^4 I_o^2 \Delta S^2}{\lambda^4} = \frac{1}{2} I_o^2 R_{rad}$	$R_{rad} = \dfrac{320\pi^4 S^2}{\lambda^4}$	
Radiation Intensity:	$P_{rad} = \int_0^{2\pi}\int_0^{\pi} \mathcal{P}_{avg}\, r^2 \sin\theta\, d\theta\, d\phi$	$\mathcal{P}_{avg} r^2 = U(\theta,\phi)$ $U_{avg} = \dfrac{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega}{4\pi} = \dfrac{P_{rad}}{4\pi}$	$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega$ $d\Omega = \sin\theta\, d\theta\, d\phi$	
Directive Gain and Directivity:	$G_d(\theta,\phi) = \dfrac{U(\theta,\phi)}{U_{avg}} = \dfrac{4\pi\, U(\theta,\phi)}{P_{rad}}$ $G_d(dB) = 10\log_{10} G_d(\theta_o,\phi_o)$	$G_d(\theta,\phi) = 1$ (isotropic radiator)	$D = [G_d(\theta,\phi)]_{\max} = \dfrac{[U(\theta,\phi)]_{\max}}{U_{avg}} = \dfrac{4\pi\,[U(\theta,\phi)]_{\max}}{P_{rad}}$ $D(dB) = 10\log_{10} D$	
Power Gain:	$G_p(\theta,\phi) = \dfrac{4\pi\, U(\theta,\phi)}{P_{in}}$	$P_{in} = P_{rad} + P_{loss}$		
Half-power point and Nulls:	Half-power point: $U(\theta) = 0.5$ HPBW: $2\theta \text{ at which } U(\theta) = 0.5$	Null points: $U(\theta) = 0$ FNBW: $2\theta \text{ at which } U(\theta) = 0$		
Antenna Field Regions:	Reactive near field: $r < 0.62\sqrt{D^3/\lambda}$	Radiating near field (Fresnel): $0.62\sqrt{D^3/\lambda} < r < 2D^2/\lambda$	Far field (Fraunhofer): $r > 2D^2/\lambda$	

Pattern Multiplicatio n Theorem:	$Array\ pattern = Array\ element\ pattern \times Array\ factor\ (\mathrm{AF})$
Two element array of Hertzian dipoles:	$r_1 \approx r + \frac{d}{2}\cos\theta$ $r_2 \approx r - \frac{d}{2}\cos\theta$ $E_{\theta 1} \approx j\eta \frac{kI_o e^{j0}\Delta l}{4\pi r}\sin\theta\, e^{-jk[r+(d/2)\cos\theta]}$ $E_{\theta 2} \approx j\eta \frac{kI_o e^{j\alpha}\Delta l}{4\pi r}\sin\theta\, e^{-jk[r-(d/2)\cos\theta]}$ $E_{\theta 1} = E_{\theta 1} + E_{\theta 1} \approx \underbrace{j\eta \frac{kI_o\Delta l}{4\pi r}\sin\theta\, e^{-jkr}}_{\text{Element Pattern (infinitesimal dipole at the origin)}}\underbrace{\left[e^{-jk(d/2)\cos\theta} + e^{j\alpha}e^{jk(d/2)\cos\theta}\right]}_{\text{Array Factor }(AF)}$ $AF = 2e^{j\frac{\alpha}{2}}\cos\left[\frac{1}{2}(kd\cos\theta + \alpha)\right]$ α: phase difference between elements
General N-Element Linear Array:	$r_1 = r$ $r_2 \approx r - d\cos\theta$ $r_3 \approx r - 2d\cos\theta$ $\vdots$ $r_N \approx r - (N-1)d\cos\theta$ $I_1 = I_o \quad I_2 = I_o e^{j\alpha} \quad I_3 = I_o e^{j2\alpha} \quad - \quad I_N = I_o e^{j(N-1)\alpha}$ N-element array centered at origin: $AF = \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$ Normalized: $(AF)_n = \frac{1}{N}\frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$ $E_{\theta 1} \approx I_o \frac{e^{-jkr}}{4\pi r} = E_o$ $E_{\theta 2} \approx I_o e^{j\alpha}\frac{e^{-jk(r-d\cos\theta)}}{4\pi r} = E_o e^{j(\alpha + kd\cos\theta)}$ $E_{\theta 3} \approx I_o e^{j2\alpha}\frac{e^{-jk(r-2d\cos\theta)}}{4\pi r} = E_o e^{j2(\alpha + kd\cos\theta)}$ $\vdots$ $E_{\theta N} \approx I_o e^{j(N-1)\alpha}\frac{e^{-jk[r-(N-1)d\cos\theta]}}{4\pi r} = E_o e^{j(N-1)(\alpha + kd\cos\theta)}$ $E_\theta = E_{\theta 1} + E_{\theta 2} + E_{\theta 3} + \cdots + E_{\theta N}$ $= E_o\left[1 + e^{j(\alpha + kd\cos\theta)} + e^{j2(\alpha + kd\cos\theta)} + \ldots + e^{j(N-1)(\alpha + kd\cos\theta)}\right]$ $= E_o[AF]$ $AF = \left[1 + e^{j(\alpha + kd\cos\theta)} + e^{j2(\alpha + kd\cos\theta)} + \ldots + e^{j(N-1)[\alpha + kd\cos\theta]}\right]$ $= \left[1 + e^{j\psi} + e^{j2\psi} + \ldots + e^{j(N-1)\psi}\right] \qquad (\psi = \alpha + kd\cos\theta)$ $= \sum_{n=1}^{N} e^{j(n-1)\psi}$
Array function:	Nulls: $\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\alpha \pm \frac{2n\pi}{N}\right)\right] \qquad n = 1, 2, 3, \ldots \quad n \neq 0, N, 2N, 3N, \ldots$ Peaks: $\theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-\alpha \pm 2m\pi)\right] \qquad m = 0, 1, 2, \ldots$
Effective area:	$A_e = \frac{P_r}{\wp_{avg}}$ $P_r = \frac{1}{2}\frac{\lvert V_{Ls}\rvert^2}{R_{rad}} = \frac{1}{2}\frac{(\lvert V_{As}\rvert/2)^2}{R_{rad}} = \frac{\lvert V_{As}\rvert^2}{8R_{rad}}$ $A_e = D\frac{\lambda^2}{4\pi}$ Effective area for any orientation: $A_e(\theta,\phi) = G_d(\theta,\phi)\frac{\lambda^2}{4\pi}$ For an infinitesimal dipole: $A_e = \frac{P_r}{\wp_{avg}} = \frac{\frac{\lvert E_s\rvert^2\lambda^2}{640\pi^2}}{\frac{\lvert E_s\rvert^2}{240\pi}} = \frac{3\lambda^2}{8\pi} = \frac{3}{2}\frac{\lambda^2}{4\pi}$ (↑ $\frac{3}{2}$: Infinitesimal dipole directivity)
Friis Transmission Formula:	$\wp_{avg} = \frac{P_t}{4\pi r^2}G_{dt}$ $P_r = \wp_{avg}A_{er}$ $P_r = G_{dt}G_{dr}\left(\frac{\lambda}{4\pi r}\right)^2 P_t$
Radar:	$P_r = \frac{\sigma\lambda^2 G_{dr}G_{dt}}{(4\pi)^3 r_1^2 r_2^2}P_{rad}$ r_1: distance from Tx to object r_2: distance from object to Rx σ: radar cross section (RCS) of object For a monostatic radar: Tx = Rx, $r_1 = r_2 = r$ $r = \left[\frac{\sigma\lambda^2 G_{dr}G_{dt}}{(4\pi)^3}\frac{P_{rad}}{P_r}\right]^{1/4}$

Radio Wave Propagation:

Free space path loss:

$$L_0 = \left(\frac{\lambda}{4\pi d}\right)^2$$

Multipath from a flat ground:

$$\Delta R = \underbrace{(R_1 + R_2)}_{\text{REFLECTED}} - \underbrace{R_o}_{\text{DIRECT}}$$

$$\Gamma = \rho e^{j\phi_\Gamma}$$

If $\psi \approx 0$

Then $\Gamma \approx -1$

$$|E_{tot}| = \underbrace{E_{ref}}_{\text{REFLECTED}} + \underbrace{E_{dir}}_{\text{DIRECT}}$$

$$= \left| f_t(\theta_A) f_r(\theta_C) \frac{e^{-jkR_o}}{4\pi R_o} \overbrace{\left[1 + \Gamma \frac{f_t(\theta_B) f_r(\theta_D)}{f_t(\theta_A) f_r(\theta_C)} e^{-jk\Delta R} \right]}^{\equiv F} \right|$$

$0 \leq F \leq 2$

$$\Delta R \approx \frac{2h_r h_t}{d}$$

$$|F| = 2\left|\sin\left(\frac{kh_t h_r}{d}\right)\right|$$

$$P_r = P_{dir}|F|^2 = P_{dir} 4\sin^2\left(\frac{kh_t h_r}{d}\right)$$

$$P_r \approx P_{dir} 4\left(\frac{kh_t h_r}{d}\right)^2$$

Maxima and minima of PPF:

Minima

$$kh_t \tan\psi = n\pi \quad (n = 0, 1, \ldots, \infty)$$

$$\frac{2\pi}{\lambda} h_t \tan\psi = n\pi$$

$$\tan\psi = n\lambda / h_t$$

Maxima

$$kh_t \tan\psi = m\pi/2 \quad (m = 1, 3, 5, \ldots, \infty)$$

$$\frac{2\pi}{\lambda} h_t \tan\psi = \frac{2n+1}{2}\pi \quad (n = 0, 1, \ldots, \infty)$$

$$\tan\psi = \frac{(2n+1)\lambda}{4h_t}$$

Coverage plot:

$$F = 2\frac{d_0}{d}\left|\sin\left(\frac{2\pi h_t h_r}{\lambda d}\right)\right| = m$$

Receiver height (h_t) plotted against range (d)

Power received:

$$Pr = 4PtGtGr\left(\frac{\lambda}{4\pi d}\right)^2 \sin^2\left(\frac{2\pi hthr}{\lambda d}\right)$$

$$P_r = P_t G_t G_r 4\left(\frac{h_t h_r}{d^2}\right)^2$$

$\sin\theta \approx \theta$ Since ht, hr <<d, θ is small

Atmospheric refraction:

Ray trajectory:

$$n R_e \sin\theta = \text{CONSTANT}$$

R_e: actual Earth radius (6378 km),

n: index of refraction, found by:

$n = \sqrt{\varepsilon_r}$ is found by:

1) $n = 1 + \chi\rho/\rho_{SL}$ + HUMIDITY TERM

χ: Gladstone-Dale constant (≈ 0.00029)

ρ, ρ_{SL}: mass densities at altitude and sea level respectively

2) $n = \frac{77.6}{T}(p + 4{,}810 e/T)10^{-6} - 1$

p: air pressure (millibars)

T: temperature (K)

e: partial pressure of water vapor (millibars)

Exact distance from Tx to horizon:

$$R_t = \sqrt{(R'_e + h_t)^2 - (R'_e)^2}$$

$\because R'_e >> h_t$

$\rightarrow$ $R_t \approx \sqrt{2R'_e h_t} \wedge R_r \approx \sqrt{2R'_e h_r}$

$\therefore$

$$R_{RH} \approx \sqrt{2R'_e h_t} + \sqrt{2R'_e h_r}$$

(distance to Radio Horizon)

Equivalent Earth radius:

Define $\kappa = \left[1 + \frac{R_e}{\sqrt{\varepsilon_o}} \frac{d}{dh}\sqrt{\varepsilon(h)}\right]^{-1}$ s.t., $[\psi(h)]^2 \approx \psi_o^2 + \frac{2h}{\kappa R_e} = \psi_o^2 + \frac{2h}{R'_e} \rightarrow R'_e = \kappa R_e$

Valid with the following assumptions:

1) constant index of refraction, 2) ray paths are nearly horizontal, and 3) $\sqrt{\varepsilon_h}$ vs. *h* is linear over the range of *h*

$\kappa = \frac{4}{3} \rightarrow R'_e = 8500\ km$ under normal conditions

Interference region:

Path-gain factor:

$$|F| = \left|1 + \rho e^{j\phi_\Gamma} e^{-jk\Delta R} \quad D\right|$$

D: divergence factor,
$\Delta R = R_1 + R_2 - R_o$

Approximate path-gain factor formula for interference region:

$$|F| = \left\{(1 + |\Gamma| D)^2 \quad - 4|\Gamma| D \sin^2\left[\frac{\phi_\Gamma - k\Delta R}{2}\right]\right\}^{\frac{1}{2}}$$

$$\Delta R = \frac{2h_1h_2}{d} J(S,T)$$

$$\tan\psi = \frac{h_1 + h_2}{d} K(S,T)$$

$$D = \left[1 + \frac{4S_1S_2^2T}{S(1-S_2^2)(1+T)}\right]^{-1/2}$$

$$S_1 = \frac{d_1}{\sqrt{2R'_e h_1}}, \quad S_2 = \frac{d_2}{\sqrt{2R'_e h_2}}$$

$$h_1 = min\{h_t, h_r\}$$
$$h_2 = max\{h_t, h_r\}$$

$$S = \frac{d}{\sqrt{2R'_e h_1} + \sqrt{2R'_e h_2}} = \frac{S_1T + S_2}{1+T}$$

$$T = \sqrt{h_1 / h_2} \quad (< 1 \text{ since } h_1 < h_2)$$

$$J(S,T) = (1 - S_1^2)(1 - S_2^2)$$

$$K(S,T) = \frac{(1 - S_1^2) + T^2(1 - S_2^2)}{1 + T^2}$$

$$d_1 = \frac{d}{2} + p\cos\left(\frac{\Phi + \pi}{3}\right)$$

$$\Phi = \cos^{-1}\left(\frac{2R'_e(h_1 - h_2)d}{p^3}\right)$$

$$p = \frac{2}{\sqrt{3}}\left[R'_e(h_1 + h_2) + \frac{d^2}{4}\right]^{1/2}$$

Alternate form of phase difference:

$$k\Delta R = \frac{2kh_1h_2}{d}(1 - S_1^2)(1 - S_2^2) = \nu\zeta\pi$$

$$\nu = \frac{4h_1^{3/2}}{\lambda\sqrt{2R'_e}} = \frac{h_1^{3/2}}{1030\lambda}$$

$$\zeta = \frac{h_2 / h_1}{d / d_{RH}}(1 - S_1^2)(1 - S_2^2)$$

Distance to radio horizon:

$$d_{RH} = \sqrt{2R'_e h_1}$$

Special case in which the reflection coefficient (Γ) is -1:

$$F = \left[(1 + D)^2 - 4D\cos^2\frac{k_0\Delta R}{2}\right]^{1/2} = \left[(1 + D)^2 - 4D\cos^2\left(\frac{\pi}{2}\nu\zeta\right)\right]^{1/2}$$

To plot the coverage diagram of the above special case, use the normalized coordinates (h_2/h_1) and d/d_T → $F = m\frac{d}{d_T}$, where $d_T = \sqrt{2R'_e h_1}$

Diffraction region:

For frequencies > 100 MHz:

$$F = V_1(X)U_1(Z_1)U_1(Z_2)$$

$Z_i = h_i / H \quad (i=1,2)$	$X = d/L$
$L = 2\left(\frac{(R_e')^2}{4k}\right)^{\frac{1}{3}} = 28.41\lambda^{1/3}(\text{km})$	$H = \left(\frac{R_e'}{2k^2}\right)^{\frac{1}{3}} = 47.55\lambda^{2/3}(\text{m})$
$V_1(X) = 2\sqrt{\pi X}e^{-2.02X}$	U_1 is found from tables or curves

Fresnel zones:

If the direct path's phase differs from the reflected path's phase, then

$$\Delta R = n\lambda/2 \qquad (n = 0,1,\ldots)$$

Assuming $\Gamma = \rho e^{j\pi} = -\rho$, then:

If **n** is **even**: two paths are out of phase and received signal is **min.**	If **n** is **even**: two paths are in phase and received signal is **max.**

$$\because R_0 \approx d \rightarrow R_0 \approx d_t + d_r = d \quad \therefore R_1 + R_2 = d + n\lambda/2$$

Radius of nth Fresnel zone:

$$r_n = \sqrt{\frac{n\lambda d_t d_r}{d}}$$

To find Tx and Rx locations and heights that give signal maxima:
1) reflections points should not lie on even Fresnel zones (if n even, received signal is min)
2) LOS should clear all obstacles by $0.6r_1$ to essentially give free space transmission.

Diffraction:

Clearance height:

$$h_c = \left(\frac{d_2h_1 + d_1h_2}{d} - h\right)\cos\theta_c$$

Diffraction loss factor F_d: (negligible if $H_c > 0.8$)

$$F_d = \left|\frac{E_{tot}}{E_0}\right| = \frac{1}{\sqrt{2}}\left|\int_{-H_c}^{\infty} e^{j\pi u^2/2}du\right|$$

where H_c is the clearance height parameter:

$$H_c \approx \sqrt{\frac{2d}{\lambda_0 d_1 d_2}}h_c$$

$$F_d(dB) = \begin{cases} 0, & H_c \geq 1 \\ 20\log(0.5 + 0.62H_c), & 0 \leq H_c \leq 1 \\ 20\log(0.5\exp(0.95H_c)), & -1 \leq H_c \leq 0 \\ 20\log\left(0.4 + \sqrt{0.1184 - (0.38 + 0.1H_c)^2}\right), & -2.4 \leq H_c \leq -1 \\ 20\log\left(\frac{-0.225}{H_c}\right), & H_c < -2.4 \end{cases}$$

Received power:

$$P_r = P_{r0}|F_d|^2$$

Earth's bulge factor:

$$b = \frac{d_t d_r}{1.5\kappa}$$

(add the max bulge factor to $0.6r_1$ to compute the minimum antenna height)

Surface waves:	Power density at Rx: $$P_r = P_{\text{dir}} \left|2A_s\right|^2$$ P_{dir}: Power density received if free space	A_s: attenuation factor
	If $d \le 50/(f_{\text{MHz}})^{1/3}$ miles, then the following equations may be used: $$\left|A_s\right| = \frac{2+0.3p}{2+p+0.6p^2} - \sqrt{p/2}\, e^{-0.6p} \sin b \quad (b \le 90^\circ)$$	$$p = \frac{kd}{2\sqrt{\varepsilon_r^2 + (\sigma/\omega\varepsilon_o)^2}} \quad \text{(numerical distance)}$$
	$$b = \tan^{-1}\left(\frac{\varepsilon_r \varepsilon_o \omega}{\sigma}\right)$$	$$\sigma/\omega\varepsilon_o = \frac{1.8 \times 10^4 \sigma}{f_{\text{MHz}}}$$
Ionospheric propagation:	Relative dielectric constant of an ionized gas (assuming electrons only): $$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}$$ v: collisions per second	N_e: electron density (m^{-3})
	$e = 1.59 \times 10^{-19}$ C, electron charge	$m = 9.0 \times 10^{-31}$ kg, electron mass
	Plasma frequency (rad/s): $$\omega_p = \sqrt{\frac{N_e e^2}{m\varepsilon_o}}$$	If no collisions occur (v=0), then the propagation constant is: $$k_c = \omega\sqrt{\mu_o \varepsilon_r \varepsilon_o} = k_o\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$ where $k_o = \omega\sqrt{\mu_o \varepsilon_o}$
	Possible cases for ω: 1) $\omega > \omega_p$: k_c is real, and the wave continues propagating since $e^{-jk_c z} = e^{-j\|k_c\|z}$ 2) $\omega < \omega_p$: k_c is imaginary, and the wave is evanescent (attenuates only) since $e^{-jk_c z} = e^{-\|k_c\|z}$ 3) $\omega = \omega_p$: $k_c = 0$, and the wave is reflected. (critical frequency)	
	At normal incidence, max frequency at which reflection occurs: $$f_c = \frac{\omega_c}{2\pi} \approx 9\sqrt{N_{e\max}}$$ Reflection can occur at smaller N_e (or higher f_c) if incidence is oblique.	
	At oblique incidence: (angle of incidence: ψ_i) $$\because\ \sin\psi_i = \sqrt{\varepsilon_r(z)} \quad \rightarrow \quad 1 - \cos^2\psi_i = 1 - \frac{81N_{e\max}}{f^2} \quad \therefore\ f_{\max} = \sqrt{\frac{81N_{e\max}}{\cos^2\psi_i}}$$	
	Maximum Usable Frequency: (from secant/Martyn's law) $$f = 9\sqrt{N_{e\max}}\sec\psi_i = f_c \sec\psi_i$$ MUF is found when $sec(\psi_i)$ is max, Optimum Usable Frequency is 0.5 to 0.8 of MUF	
	Skip distance: Flat Earth model: $$d = 2h' \cdot tan(\psi_i)$$	Spherical Earth model: $$d = 2\sqrt{2R'_e h'}$$

<table>
<tr><td rowspan="6">Ionospheric propagation (continued):</td><td colspan="2">Referring to Figure 1 below, curved Earth equations:
$\frac{1 + h'/R_e' - \cos\theta}{\sin\theta} = \frac{1}{\tan\psi_i}$ $\theta = \frac{d}{2R_e'}$ $R = \frac{2R_e' \sin\theta}{\sin\psi_i}$</td></tr>
<tr><td colspan="2">Launch angle: $\Delta = \phi - 90^\circ = 90^\circ - \theta - \psi_i$</td></tr>
<tr><td colspan="2">Modified Friis formula:
$$P_r = \frac{P_t G_t G_r}{(4\pi R/\lambda)^2} L_x L_\alpha$$</td></tr>
<tr><td>The losses in dB are: (negative)
$L_x = L_{\text{pol}} + L_{\text{refl}} - G_{\text{iono}}$</td><td>$G_{iono}$: gain due to focusing by the curvature of the ionosphere</td></tr>
<tr><td>L_{refl}: reflection loss in case of multiple hops</td><td>L_α: absorption loss</td></tr>
<tr><td>L_{pol}: polarization loss due to Faraday rotation and earth reflections</td><td>R: great circle path via the virtual reflection point (as defined previously)</td></tr>
<tr><td rowspan="3">Ducts and nonstandard refraction: (refer to Table 1 below)</td><td colspan="2">Refractivity:
$N(h) = [n(h) - 1]10^6$ ∧ $n(h) = \sqrt{\varepsilon_r(h)}$</td></tr>
<tr><td>Normal atmosphere (equivalent earth radius): Gradient of the vertical refractive index is linear if
$dN/dh \approx -39$ N units/km</td><td>If
$dN/dh < -157$
Rays will return to the surface</td></tr>
<tr><td>Modified refractivity:
$M(h) = N(h) + 10^6 (h/R_e')$</td><td>For ducting:
$dM/dh = dN/dh + 157$</td></tr>
<tr><td rowspan="3">Attenuation due to rain and gases:</td><td colspan="2">attenuation in dB/km:
$$A = aR^b$$
where R is the rate of rain in mm/hr</td></tr>
<tr><td>$a = G_a f_{\text{GHz}}^{E_a}$</td><td>$b = G_b f_{\text{GHz}}^{E_b}$</td></tr>
<tr><td>$G_a = 6.39 \times 10^{-5}$ $E_a = 2.03$ $f_{\text{GHz}} < 2.9$
$= 4.21 \times 10^{-5}$ $= 2.42$ $2.9 \le f_{\text{GHz}} < 54$
$= 4.09 \times 10^{-2}$ $= 0.699$ $54 \le f_{\text{GHz}} < 180$</td><td>$G_b = 0.851$ $E_b = 0.158$ $f_{\text{GHz}} < 8.5$
$= 1.41$ $= -0.0779$ $8.5 \le f_{\text{GHz}} < 25$
$= 2.63$ $= -0.272$ $25 \le f_{\text{GHz}} < 164$</td></tr>
</table>

Conversions: 0.0254 m = 1 in; 12 in = 1 ft; 3.3 ft = 1 m; 5280 ft = 1 mi; 1 km = 0.62 mi

Okumura model:	Valid for: $150\text{MHz} \le f_c \le 1920\text{MHz}$, $1\text{km} \le d \le 100\text{km}$, and $30\text{m} < h_{te} < 1000\text{m}$	
	$L_{50}(dB) = L_F + A_{mu}(f,d) - G(h_{te}) - G(h_{re}) - G_{AREA}$	
	L_{50}: 50% of propagation path loss	L_F: free space propagation loss
	$A_{mu}(f,d)$: median attenuation relative to free space, found from plots	$G(h_{te})$: base station antenna height gain factor
	$G(h_{re})$: mobile antenna height gain factor	G_{AREA}: gain due to environment, found from plots
	Gain factor calculations: $G(h_{te}) = 20\log\left(\frac{h_{te}}{200}\right) \quad 1000m > h_{te} > 30m$ $G(h_{re}) = 10\log\left(\frac{h_{re}}{3}\right) \quad h_{re} \le 3m$ $G(h_{re}) = 20\log\left(\frac{h_{re}}{3}\right) \quad 10m > h_{re} > 3m$	
	$L_F(dB) = 32.45 + 20\log_{10} f_c(MHz) + 20\log_{10} d(km)$	
	$P_r = EIRP + G_r - L_{50}$	where the Equivalent Isotropic Radiated Power, $EIRP = P_t G_t$
Hata propagation model:	Path loss for different terrains: $L_{50-urban}(dB) = 69.55 + 26.16\log f_c - 13.82\log h_{te} - \alpha(h_{re}) + (44.9 - 6.55\log h_{te})\log d$ $L_{50-suburban}(dB) = L_{50-urban} - 2\left[\log\left(f_c/28\right)\right]^2 - 5.4$ $L_{50-rural}(dB) = L_{50-urban} - 4.78\left[\log f_c\right]^2 - 18.33\log f_c - 40.94$	
	f_c: frequency (in MHz) from 150MHz to 1500MHz	h_{te}: effective Tx antenna height (in m) from 30m to 200m
	h_{re}: Rx antenna height (in m) from 1m to 10m	d: distance from Tx to Rx (in km)
	$\alpha(h_{re})$ is the correction factor for effective mobile height, calculated from: 1) for a small city: $\alpha(h_{re}) = (1.1\log f_c - 0.7)h_{re} - (1.56\log f_c - 0.8)dB$ 2) for a large city: $\alpha(h_{re}) = 8.29(\log 1.54h_{re})^2 - 1.1dB \quad \text{for} \quad f_c \le 300MHz$ $\alpha(h_{re}) = 3.2(\log 11.75h_{re})^2 - 4.97dB \quad \text{for} \quad f_c \ge 300MHz$	

<table>
<tr><td rowspan="2">Environment-based path loss:</td><td colspan="2">$$L_p = L_0 + 10\alpha \log_{10}(d)$$</td></tr>
<tr><td>L_0: frequency dependent component, independent of distance. Often path loss at 1m</td><td>α: path loss gradient or path loss exponent</td></tr>
<tr><td rowspan="2">General path loss formula:</td><td colspan="2">$$P_r(d) \propto \left(\frac{P_t}{d^\alpha}\right) \text{ OR } P_r(d) = \left(\frac{P_t}{L_0(d/d_0)^\alpha}\right)$$</td></tr>
<tr><td>α: path loss gradient or path loss exponent, as in environment-based path loss</td><td>L_0: constant computed at distance d_0
d_0 is 1m in indoor areas and 100m or 1km in outdoor areas</td></tr>
<tr><td rowspan="2">Indoor path loss models:</td><td colspan="2">Motley-Keenan and Rappaport model: (L_0 as above)
$$L_p = L_0 + 20\log d + \sum_i m_i W_i + \sum_j n_j F_j$$</td></tr>
<tr><td>m_i: number of partitions of type i,
W_i: loss associated with partition i</td><td>n_i: number of floors of type j,
F_i: loss associated with floor j</td></tr>
<tr><td>Skin depth:</td><td colspan="2">$$\delta_S = \left[\frac{k^2}{2}\left[\sqrt{{\varepsilon_r}^2 + \left[\sqrt{\frac{\sigma}{\omega\varepsilon_0}}\right]^2} - \varepsilon_r\right]\right]^{-1/2}$$
Recall from EM2: k = 2𝝅/λ and ε_r and $\sigma/\omega\varepsilon_0$ are the real and imaginary parts of the dielectric constant respectively</td></tr>
<tr><td rowspan="4">Specific Absorption Rate:</td><td colspan="2">In terms of energy (W), time (t), and mass (m) or volume (V) and density (ρ in kg/m^3):
$$\text{SAR} = \frac{d}{dt}\frac{dW}{dm} = \frac{d}{dt}\frac{dW}{\rho dV}$$</td></tr>
<tr><td colspan="2">In terms of RMS electric field (E in V/m), conductivity (σ in S/m), and density:
$$\text{SAR} = \frac{\sigma E^2}{\rho}$$</td></tr>
<tr><td colspan="2">As a function of rate of rise in temperature:
$$SAR = \left.\frac{c\,\Delta T}{\Delta t}\right|_{t=0}$$</td></tr>
<tr><td colspan="2">ΔT: change in temperature (°C or K) | Δt: duration of exposure (s) | C: specific heat capacity [J/(kg °C)]</td></tr>
</table>

<table>
<tr><td rowspan="5">Compliance with RF exposure standards:</td><td>∵
$$E = \sqrt{\left|E_x^2\right| + \left|E_y^2\right| + \left|E_z^2\right|}$$</td><td>∵
$$H = \sqrt{\left|H_x^2\right| + \left|H_y^2\right| + \left|H_z^2\right|}$$</td></tr>
<tr><td>↪ $S_E = 0.1\left[\left|E_x\right|^2 + \left|E_y\right|^2 + \left|E_z\right|^2\right] / 376.73$</td><td>↪ $S_H = 0.1\left[\left|H_x\right|^2 + \left|H_y\right|^2 + \left|H_z\right|^2\right]376.73$</td></tr>
<tr><td colspan="2">where S_E and S_H are in mW/cm^2, and $\eta_0 = 120\pi = 376.73$ (intrinsic impedance)</td></tr>
<tr><td colspan="2">∴
$$S = \max\{S_E, S_H\}$$</td></tr>
<tr><td colspan="2">Max combined power densities in environments with mixed frequencies sources:
$$\sum_{i=1}^{n} \frac{S_i}{L_i} = \frac{S_1}{L_1} + \frac{S_2}{L_2} + \ldots + \frac{S_n}{L_n} \leq 1$$
where S_i is the power density at frequency band i, and L_i is the Maximum Permissible Exposure (MPE) at frequency band i.</td></tr>
</table>

Figures and Tables:

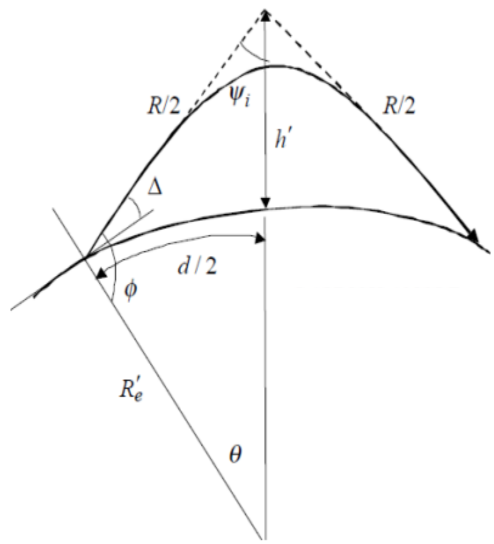

Figure 1: Ionospheric propagation angles and sides.

dN/dh	Ray Curvature	κ	Atmospheric Refraction	Virtual Earth	Horizontally Launched Ray
> 0	up	< 1	below normal	more convex	moves away from Earth
0	none	1		actual	
$0 > \frac{dN}{dh} > -39$	down	> 1		less convex	
-39		4/3	normal		
$-39 > \frac{dN}{dh} > -157$		$> 4/3$	above normal		
-157				plane	parallel to Earth
$<$ -157			super-refraction	concave	draws closer to Earth

Table 1: Refractivity and ducting conditions.

Frequency Range, *f* (MHz)	**Electric Field Strength, *E* (V/m)**	**Magnetic Field Strength, *H* (A/m)**	**Power Density *E* Field; *H* Field *S* (mW/cm²)**	**Averaging Time $\|E^2\|$; $\|H^2\|$; *S* (min.)**
0.003–0.1	614	163	(100; 1,000,000) *	6
0.1–3.0	614	$16.3/f$	$(100; 10{,}000/f^2)$ *	6
3.0–30	$1{,}824/f$	$16.3/f$	$(900/f^2; 10{,}000/f^2)$ *	6
30–100	61.4	$16.3/f$	$(1.0; 10{,}000/f^2)$ *	6
100–300	61.4	0.163	1.0	6
300–3,000	—	—	$f/300$	6
3,000–15,000	—	—	10	6
15,000–300,000	—	—	10	$616{,}000/f^{1.2}$

Table 2: ANSI/IEEE C95.1-1992 Radio Protection Guidelines for Controlled Environments.

* Plane-wave equivalent power density, not suitable for near-field conditions but can be used for comparisons.

Frequency Range, f (MHz)	Electric Field Strength, E (V/m)	Magnetic Field Strength, H (A/m)	Power Density E Field; H Field S (mW/cm^2)	Averaging Time $\|E^2\|$; S (min.)	Averaging Time $\|H^2\|$; S (min.)
0.003–0.1	614	163	(100; 1,000,000) *	6	6
0.1–1.34	614	$16.3/f$	$(100; 10{,}000/f^2)$ *	6	6
1.34–3.0	$823.8/f$	$16.3/f$	$(180/f^2; 10{,}000/f^2)$ *	$f^2/0.3$	6
3.0–30	$823.8/f$	$16.3/f$	$(180/f^2; 10{,}000/f^2)$ *	30	6
30–100	27.5	$158.3/f^{1.668}$	$(0.2; 940{,}000/f^{3.336})$ *	30	$0.0636 f^{1.337}$
100–300	27.5	0.0729	0.2	30	30
300–3,000	—	—	$f/1{,}500$	30	—
3,000–15,000	—	—	$f/1{,}500$	$90{,}000/f$	—
15,000–300,000	—	—	10	$616{,}000/f^{1.2}$	—

Table 3: ANSI/IEEE C95.1-1992 Radio Frequency Protection Guides for Uncontrolled Environments.

* Plane-wave equivalent power density, not suitable for near-field conditions but can be used for comparisons.

Frequency range	Magnetic flux density (mT)	Current density (mA/m^2) (rms)	Whole body average SAR (W/kg)	Localised SAR (head and trunk) (W/kg)	Localised SAR (limbs) (W/kg)	Power density, S (W/m^2)
0 Hz	40	—	—	—	—	—
>0-1 Hz	—	8	—	—	—	—
1-4 Hz	—	8/f	—	—	—	—
4-1 000 Hz	—	2	—	—	—	—
1 000 Hz-100 kHz	—	f/500	—	—	—	—
100 kHz-10 MHz	—	f/500	0,08	2	4	—
10 MHz-10 GHz	—	—	0,08	2	4	—
10-300 GHz	—	—	—	—	—	10

Table 4: Basic restrictions for electric, magnetic, and electromagnetic fields.

Frequency Range, f (MHz)	Electric Field Strength, E (V/m)	Magnetic Field Strength, H (A/m)	Power Density S (mW/cm^2)	Averaging Time (min.)
0.3–3.0	614	$1.63/f$	(100)*	6
3.0–30	$1,842/f$	$4.89/f$	$(900/f^2)$*	6
30–300	61.4	0.163	1.0	6
300–1,500	—	—	$f/300$	6
1,500–100,000	—	—	5	6

Table 5: 1996 FCC limits for occupational/controlled environments.

Frequency Range, f (MHz)	Electric Field Strength, E (V/m)	Magnetic Field Strength, H (A/m)	Power Density S (mW/cm^2)	Averaging Time (min.)
0.3–1.34	614	$1.63/f$	(100)*	30
01.34–30	$824/f$	$2.19/f$	$(180/f^2)$*	30
30–300	27.5	0.073	0.2	30
300–1,500	—	—	$f/1,500$	30
1,500–100,000	—	—	1.0	30

Table 6: 1996 FCC limits for general population/uncontrolled environments.