Mid:

Floating Point Representation:

$$m \cdot b^{e}$$

m: mantissa
b: base (2 for binary)
e: exponent

Condition: $\frac{1}{b} \leq m < 1$

Chopping:
$$\frac{|\Delta x|}{|x|} \leq \mathscr{E}$$

Rounding:
$$\frac{|\Delta x|}{|x|} \leq \frac{\mathscr{E}}{2}$$

$|\Delta x|$: interval between numbers

Machine Epsilon:
$$\mathscr{E} = b^{1-t}$$

t: number of significant figures in mantissa

Taylor Series Approximation:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n$$

h: step size
$$\boldsymbol{h = x_{i+1} - x_i}$$

Remainder (absolute error): $\boldsymbol{O(h^{n+1})}$
$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$
where $\xi \in [\boldsymbol{x_i},\ \boldsymbol{x_{i+1}}]$

Error Equations:

Percentage True Relative Error:
$$\varepsilon_t = \left|\frac{x_r{}^{true} - x_r{}^{approximate}}{x_r{}^{true}}\right| 100\%$$

Relative Percentage Approximate Error:
$$\varepsilon_a = \left|\frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}}\right| 100\%$$

Finite Difference Approximations of first derivative:

Backward:
$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{h}$$

Forward:
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i)$$

Centered:
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

Total Numerical Error: (for a centered finite difference approximation)

$$\underbrace{f'(x_i)}_{\text{True value}} = \underbrace{\frac{\tilde{f}(x_{i+1}) - \tilde{f}(x_{i-1})}{2h}}_{\text{Finite-difference approximation}} + \underbrace{\frac{e_{i+1} - e_{i-1}}{2h}}_{\text{Round-off error}} - \underbrace{\frac{f^{(3)}(\xi)}{6}h^2}_{\text{Truncation error}}$$

Where $\boldsymbol{e_{i\mp1}}$ is the roundoff error associated with the function of the lower/upper x as follows:
$$f(x_{i-1}) = \tilde{f}(x_{i-1}) + e_{i-1}$$
$$f(x_{i+1}) = \tilde{f}(x_{i+1}) + e_{i+1}$$

If $\boldsymbol{e_{i\mp1}} \leq \boldsymbol{\varepsilon}$ (machine epsilon), and $\boldsymbol{f^{(3)}(\xi)} \leq \boldsymbol{M}$, then:
$$\text{Total error} = \left|f'(x_i) - \frac{\tilde{f}(x_{i+1}) - \tilde{f}(x_{i-1})}{2h}\right| \leq \frac{\varepsilon}{h} + \frac{h^2 M}{6}$$

Optimal Step Size:
$$h_{\text{opt}} = \sqrt[3]{\frac{3\varepsilon}{M}}$$

Newton-Raphson: (modified for multiple roots)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Known multiplicity (m):
$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

Unknown multiplicity:
$$x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$

Secant Method:	$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$	Newton-Raphson with $f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$
	Modified Secant Method (with small perturbation factor δ): $$x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$	
Nonlinear Simultaneous Equations:	$$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}}$$	$$y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}}$$
	$$\begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix} \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix} = \begin{bmatrix} -u_i \\ -v_i \end{bmatrix}$$	
Bisection Method:	$$x_r = \frac{x_l + x_u}{2}$$	if $f(x_r) \cdot f(x_l) < 0$, set new $x_u = x_r$ and repeat if $f(x_r) \cdot f(x_l) > 0$, set new $x_l = x_r$ and repeat
Error Equations for Bisection:	Percentage Approximate Error: $$\varepsilon_a = \left\| \frac{x_u - x_l}{x_u + x_l} \right\| 100\%$$	Absolute Error: $$E_a^n = \frac{\Delta x^0}{2^n}$$ n: iteration number $\Delta x^0 = {x_u}^0 - {x_l}^0$ Δx^0: upper value of initial interval minus lower value of initial interval
False Position Method: (linear interpolation)	$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$	if $f(x_r) \cdot f(x_l) < 0$, set new $x_u = x_r$ and repeat if $f(x_r) \cdot f(x_l) > 0$, set new $x_l = x_r$ and repeat
Gauss-Seidel-Jacobi for 3x3 matrix:	$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$	$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$ $x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$ $x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$ Conditions for Convergence: $\|a_{ii}\| > \sum_{j=1, j \neq i}^{n} \|a_{ij}\|$ Magnitude of the diagonal element must be larger than the sum of the magnitudes of the other elements in its row.

Final:

Best-fit Criterion:	$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2$ $S_r = \sum_{i=1}^{n} (y_i - f(x_i))^2$
Linear Regression:	$y = a_0 + a_1 x + e$ $S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$ $\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma {x_i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i x_i \end{bmatrix}$ For minimum Sr: $a_0 = \bar{y} - a_1 \bar{x}$ s.t. $\bar{y}$ & $\bar{x}$ are the mean values. $a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$ Standard error of the estimate: $s_{y/x} = \sqrt{\frac{S_r}{n-2}}$ Coefficient of Determination: s.t. $S_t = \Sigma(y_i - \bar{y})^2$ $r^2 = \frac{S_t - S_r}{S_t}$ $r = \frac{n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)}{\sqrt{n\Sigma x_i^2 - (\Sigma x_i)^2}\sqrt{n\Sigma y_i^2 - (\Sigma y_i)^2}}$
Linearizing Common Relations:	Exponential: $y = \alpha_1 e^{\beta_1 x} \rightarrow$ $\ln y = \ln \alpha_1 + \beta_1 x$ Power: $y = \alpha_2 x^{\beta_2} \rightarrow$ $\log y = \beta_2 \log x + \log \alpha_2$ Saturation Growth Rate: $y = \alpha_3 \frac{x}{\beta_3 + x} \rightarrow \frac{1}{y} = \frac{\beta_3}{\alpha_3}\frac{1}{x} + \frac{1}{\alpha_3}$
Polynomial Regression:	$y = a_0 + a_1 x + a_2 x^2 + e$ $S_r = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2\right)^2$ $\begin{bmatrix} n & \Sigma x_i & \Sigma {x_i}^2 \\ \Sigma x_i & \Sigma {x_i}^2 & \Sigma {x_i}^3 \\ \Sigma {x_i}^2 & \Sigma {x_i}^3 & \Sigma {x_i}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma y_i x_i \\ \Sigma y_i {x_i}^2 \end{bmatrix}$ $s_{y/x} = \sqrt{\frac{S_r}{n - (m+1)}}$ $r = \frac{\Sigma(y_i - \bar{y})^2 - S_r}{\Sigma(y_i - \bar{y})^2}$
General Linear Least Squares:	For $\{Y\} = [Z]\{A\} + \{E\}$ $\{Y\}^T = \lfloor y_1 \quad y_2 \quad \cdots \quad y_n \rfloor$ $S_r = \sum_{i=1}^{n} \left(y_i - \sum_{j=0}^{m} a_j z_{ji}\right)^2$ Coefficients: $\{A\}^T = \lfloor a_0 \quad a_1 \quad \cdots \quad a_m \rfloor$ Errors: $\{E\}^T = \lfloor e_1 \quad e_2 \quad \cdots \quad e_n \rfloor$ $\left[[Z]^T[Z]\right]\{A\} = \left\{[Z]^T\{Y\}\right\}$ $\{A\} = \left[[Z]^T[Z]\right]^{-1} \left\{[Z]^T\{Y\}\right\}$ $[Z] = \begin{bmatrix} z_{01} & z_{11} & \cdots & z_{m1} \\ z_{02} & z_{12} & \cdots & z_{m2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z_{0n} & z_{1n} & \cdots & z_{mn} \end{bmatrix}$
Nonlinear Regression:	$y_i - f(x_i)_j = \frac{\partial f(x_i)_j}{\partial a_0} \Delta a_0 + \frac{\partial f(x_i)_j}{\partial a_1} \Delta a_1 + e_i$ $\{D\} = [Z_j]\{\Delta A\} + \{E\}$ $[Z_j] = \begin{bmatrix} \partial f_1/\partial a_0 & \partial f_1/\partial a_1 \\ \partial f_2/\partial a_0 & \partial f_2/\partial a_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \partial f_n/\partial a_0 & \partial f_n/\partial a_1 \end{bmatrix}$ $\{D\} = \begin{Bmatrix} y_1 - f(x_1) \\ y_2 - f(x_2) \\ \cdot \\ \cdot \\ \cdot \\ y_n - f(x_n) \end{Bmatrix}$ $\{\Delta A\} = \begin{Bmatrix} \Delta a_0 \\ \Delta a_1 \\ \cdot \\ \cdot \\ \cdot \\ \Delta a_m \end{Bmatrix}$ $a_{0,j+1} = a_{0,j} + \Delta a_0$ $a_{1,j+1} = a_{1,j} + \Delta a_1$ $\lvert\varepsilon_a\rvert_k = \left\lvert \frac{a_{k,j+1} - a_{k,j}}{a_{k,j+1}} \right\rvert 100\%$

Linear Newton's Divided Difference:	Linear Interpolation: $f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$
Quadratic Newton's Divided Difference:	$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$ $b_0 = f(x_0)$ $b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ $b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$ ∴ $f_2(x) = a_0 + a_1 x + a_2 x^2$ s.t. $a_0 = b_0 - b_1 x_0 + b_2 x_0 x_1$ $a_1 = b_1 - b_2 x_0 - b_2 x_1$ $a_2 = b_2$
General Newton's Divided Difference:	$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$ $b_0 = f(x_0)$ $b_1 = f[x_1, x_0]$ $b_2 = f[x_2, x_1, x_0]$ . . . $b_n = f[x_n, x_{n-1}, \ldots, x_1, x_0]$ $f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$ $f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$ $f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$ $f_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0]$ $+ \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_n, x_{n-1}, \ldots, x_0]$
Errors in General Newton's:	$R_n = f_{n+1}(x) - f_n(x)$ $R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$ $R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$ $R_n = f[x, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\cdots(x - x_n)$ $R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\cdots(x - x_n)$
Lagrange Interpolating Polynomials:	$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$ $L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$ e.g., $f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$ $R_n = f[x, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i)$
Linear Splines:	$f(x) = f(x_0) + m_0(x - x_0) \quad x_0 \le x \le x_1$ $f(x) = f(x_1) + m_1(x - x_1) \quad x_1 \le x \le x_2$ . . $f(x) = f(x_{n-1}) + m_{n-1}(x - x_{n-1}) \quad x_{n-1} \le x \le x_n$ $m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$

Quadratic Splines:	$f_i(x) = a_i x^2 + b_i x + c_i$ 1. $a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$ $a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$ 2. $a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$ $a_n x_n^2 + b_n x_n + c_n = f(x_n)$ 3. $2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i$	Conditions: 1. Values of adjacent polynomials must be equal at interior knots. 2. First and last functions must pass through the first and last points respectively. 3. First derivative of adjacent functions must be equal at interior knots. 4. $a_1 = 0$.
Cubic Splines:	$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$	Conditions: same as quadratic + 4. Second derivatives at end knots must be zero. 5. Second derivatives at interior knots must be equal
Trapezoidal Rule:	Single $I = (b-a)\frac{f(a)+f(b)}{2}$	Multiple: $I = (b-a)\,\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}$
	Error in single: $E_t = -\frac{1}{12} f''(\xi)(b-a)^3$ Error in multiple: $E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$	$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$ Where the bar $\bar{\ }$ represents a mean value.
Simpson's 1/3 Rule:	Single: $I \cong (b-a)\,\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$	Multiple: $I \cong (b-a)\,\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$
	Error in single: $E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$	Error in multiple: $E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$
Simpson's 3/8 Rule:	$I \cong (b-a)\,\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$	Truncation error: $E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$
High Accuracy Forward derivatives: Error: $O(h^2)$	$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$	$f''(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$
High Accuracy Backward derivatives: Error: $O(h^2)$	$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$	$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$
High Accuracy Centered derivatives: Error: $O(h^4)$	$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$ $f''(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$	
ODE Second-order Taylor Series Method:	$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + O(h^3)$	s.t. $\frac{dy(x)}{dx} = f(y,x)$ $\frac{d^2y}{dx^2}$ must be derived from $f(y,x)$

ODE:

Method	Local truncation error	Global truncation error
Euler Method $y_{i+1} = y_i + h\ f(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor: $y_{i+1}^{0} = y_i + h\ f(x_i, y_i)$ Corrector: $y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k})\right)$	$O(h^3)$	$O(h^2)$
Midpoint $y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\ f(x_i, y_i)$ $y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$