Subject Intro Date 28/2/2021 No.

* solutions to a given problem are either:
- analytical : exact solutions / closed form
- numerical : trial & error procedures/algorithms that yield an approximate solution

- analytical solutions can generally be found for linear and simplified models.

\+ errors of numerical methods:
- modelling error : errors related to the model used, like not accounting for temperature change when measuring pressure
- numerical error : mainly roundoff and truncation errors.

- roundoff errors result from limitations in number storage while truncation errors are caused by limitations of the mathematical operations used.

\+ error definitions:

	True error		approximate error	
absolute error	(E_t)	$\lvert X_t - X_{app} \rvert$	(E_a)	$\lvert X_{current} - X_{previous} \rvert$
relative error	(RE_t)	$\left\lvert \frac{X_t - X_{app}}{X_t} \right\rvert$	(RE_a)	$(E_a) / X_{current}$
percentage relative error	$(\%RE_t)$	$(RE_t) \times 100\%$	$(\%RE_a)$	$(RE_a) \times 100\%$

where X_t : true value of X , X_{app} : approximate value of X

- True error's main disadvantage is needing to know X_t
- approximate error is iterative and each new approximation is more accurate than the previous with a decreasing increase in accuracy.

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \; x_4 \; ---$$

- accuracy is the "bias", whereas precision is the "variance".
- accuracy and precision are relative.

Subject 3.4: Round-off errors Date 2/3/2021 No.

* floating point representation: $N = \pm M \cdot b^{\pm e}$ | $\frac{1}{b} \leq M < 1$ $\boxed{\text{normalization constraint}}$

where M: mantissa, b: base, and e: exponent

+ drawbacks of floating point representation:

1- a limited range of quantities can be represented

- overflow error: occurs when employing numbers outside the acceptable range

- underflow error: occurs when attempting to represent numbers whose mantissa is smaller than the reciprocal of the base

2- a finite number of significant figures can be represented. hence, chopping or rounding-off must be done.

3- the interval between the numbers, Δx, increases with the increase of the numbers' magnitudes. implying that the quantizing error will increase as the magnitude increases:

if $\mathcal{E} = b^{1-t}$ | where $\mathcal{E}$: machine epsilon, b: base, t: significant figs of M

$\rightarrow \frac{|\Delta x|}{|x|} \leq \mathcal{E}$ if chopping was used.

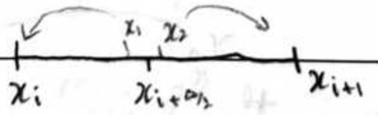
$\rightarrow \frac{|\Delta x|}{|x|} \leq \frac{\mathcal{E}}{2}$ if round-off was used.

+ Chopping : x is assumed equal to x_i

$\therefore$ e is always positive $(x - x_i)$

& $0 \leq e < (x_{i+1} - x_i)$

+ Round-off : $\begin{cases} \text{Round}(x) = x_i \text{ if } \boxed{x \leq x_{i+\Delta/2}} \\ \text{Round}(x) = x_{i+1} \text{ if } \boxed{x > x_{i+\Delta/2}} \end{cases}$



$\therefore$ e is positive or negative

& $0 \leq |e| \leq \Delta/2$

* Subtractive cancellation: Round-off errors resulting from subtracting two nearly equal floating point numbers.

- Subtractive cancellation often occurs when calculating roots of quadratic equations where $b^2 \gg 4ac$. To mitigate this, an alternative formula is used

$$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix} = \frac{-2c}{b \pm \sqrt{b^2-4ac}}}$$

- Chopping and rounding must be done along every step of the solution.

Subject Chapter 4: truncation errors & taylor series Date 13/3/2021 No.

* Truncation errors: errors resulting from using approximations instead of exact mathematical procedure.

Taylor series:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1}-x_i)^2$$
$$+ \cdots + \frac{f^{(n)}(x_i)}{n!}\cdot(x_{i+1}-x_i)^n + R_n$$

Where R_n is the remainder:

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$ t: dummy variab

or

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x_{i+1}-x_i)^{n+1}$$

where ξ is a value of x, $x_i \leq \xi \leq x_{i+1}$

- the taylor series will only be exact if (n) terms are used for an (n)th degree polynomial
- taylor series is finite if n is a finite integer, taylor series for polynomials of a fractional degree are infinite.

+ if $(x_{i+1} - x_i) = h$ then the remainder, $R_n = O(h^{n+1})$ where O is called the big O notation

hence, the error is proportional to the step size (h) raised to (n+1)

- the big O gives a rough measure of the range of R_n

+ numerical differentiation:

the three different types:
- backward finite difference approximation
- forward finite difference approximation
- central finite difference approximation

* first order approximations:

Subject Date No.

- forward finite difference:

$\because f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + R_1 \quad R_1 = O(h^2)$

$\rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{(i+1)} - x_i} - \frac{R_1}{x_{i+1} - x_i} \quad , x_{i+1} - x_i = h$

$\because \frac{O(h^2)}{h} = O(h)$

$$\rightarrow \boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)}$$

- backward finite difference:

$\because f(x_i) = f(x_{i-1}) + f'(x_{i-1}) \cdot (x_i - x_{i-1}) + R_1$

$\rightarrow f'(x_{i-1}) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} + O(h) \quad , \text{assume } \boxed{h = x_{i-1} - x_i}$

$$\rightarrow \boxed{f'(x_{i-1}) = \frac{f(x_{i-1}) - f(x_i)}{h} + O(h)}$$

- central finite difference:

$$f(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

Subject Root finding & localization Date 14/3/2021 No.

- to find values (x_r) for a function $f(x)$, where $f(x_r) = 0$, we can use two main methods:
  - + bracketing: requires two points that bracket the root
    subdivided into: bisection and false position
  - + open methods: require one or two points that don't necessarily bracket the roots.
    subdivided into: fixed point, newton raphson, secant

& bisection method: (steps)

① choose two values that yield opposite signs when substituted into the function (f):
upper value, x_u and lower value x_l → $f(x_u) \cdot f(x_l) < 0$

② the root is estimated by:

$$x_r = \frac{x_u + x_l}{2}$$

③ iterate by following this:

$$f(x_l) \cdot f(x_r) \begin{cases} < 0, & \text{the root lies in the lower subinterval} \\ & \text{set } x_u = x_r \text{ and repeat ②} \\ = 0, & \text{the root} = x_r \text{, end} \\ > 0, & \text{the root lies in the upper subinterval} \\ & \text{set } x_l = x_r \text{ and repeat ②} \end{cases}$$

example from notes: $f(x) = e^{-x} - x$, $x \in [0,1]$

take $x_u^1 = 1$, $x_l^1 = 0$ [$f(x_l) \cdot f(x_u) = -0.6321 < 0$ ✓] $\rightarrow x_r^1 = \frac{1+0}{2} = \frac{1}{2}$

$\therefore f(x_r^1) \cdot f(x_l^1) = 0.106 > 0 \rightarrow x_l^2 = x_r^1 = \frac{1}{2}$.

$\rightarrow x_r^2 = \frac{1 + \frac{1}{2}}{2} = \frac{3}{4} \rightarrow f(x_r^2) \cdot f(x_l^2) = -0.0296 < 0$

$\therefore x_u^3 = x_r^2 \rightarrow x_r^3 = \frac{3/4 + 1/2}{2} = \boxed{\frac{5}{8}}$

- the bisection method is advantageous because it is neat and allows for easy error analysis.
- for the zeroth iteration, the absolute error, $E_a^{(0)} = x_u^{(0)} - x_l^{(0)} = \Delta x^{(0)}$
- since the error is halved every iteration, the max error after (n) iterations is: $E_a^{(n)} = \frac{\Delta x^{(0)}}{2^n}$
- If E_a^d is the desired error, then $n = \log_2\left(\frac{\Delta x_0}{E_a^d}\right)$

example: assume $x_r \in [0,3]$ for a certain function $\rightarrow x_u^0 = 3 \wedge x_l^0 = 0$

$\therefore \Delta x^0 = 3$, to get an error less than 0.001 (Regardless of the function) we need to iterate n times, where n=

$\because E_a^d = 0.001 \wedge \Delta x^0 = 3 \rightarrow n = \log_2\left(\frac{3}{0.001}\right) \approx 11.55$

$\therefore$ we need to iterate 12 times.

- we do not need to know f(x) to calculate n

Subject 4.3: total numerical error (Revisited) Date 15/3/2021 No.

& total numerical error : Sum of truncation and round-off errors.

- round-off error increases due to subtractive cancellation or an increase in number of steps
- truncation error can be decreased by reducing the step size (h)
- reducing (h) can lead to subtractive cancellation.

* lower truncation error gives higher round-off error and vice versa

∵ centered difference approximation of first derivative :

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

$$\& \ O(h^2) = -\frac{f^{(3)}(\xi)}{3!} h^2$$

$$\rightarrow \underbrace{f'(x_i)}_{\text{true value}} = \underbrace{\frac{f(x_{i+1}) - f(x_{i-1})}{2h}}_{\text{finite difference approximation}} - \underbrace{\frac{f^{(3)}(\xi)}{3!} \cdot h^2}_{\text{truncation error}}$$

∵ a computer is being used, $f(x_{i+1})$ & $f(x_{i-1})$ are rounded-off

⇒ round-off error exists : $f(x_{i+1}) = \tilde{f}(x_{i+1}) + e_{i+1}$

$$f(x_{i-1}) = \tilde{f}(x_{i-1}) + e_{i-1}$$

∴

$$\underbrace{f'(x_i)}_{\text{true value}} = \underbrace{\frac{\tilde{f}(x_{i+1}) - \tilde{f}(x_{i-1})}{2h}}_{\text{finite difference approximation}} + \underbrace{\frac{e_{i+1} - e_{i-1}}{2h}}_{\text{round-off error}} - \underbrace{\frac{f^{(3)}(\xi)}{3!} \cdot h^2}_{\text{truncation error}}$$

→ round-off error $\propto 1/h$ (decreases as h increases)

→ truncation error $\propto h^2$ (increases as h^2 increases)

- if each component of the round-off error has an upper-bound equal to the machine epsilon $(\mathcal{E})$ then $(e_{i+1} - e_{i-1})_{max} = 2\mathcal{E}$
  & if $f_f^{(3)}(\xi)$ has a max value (M)
  then the max total error:

$$\frac{\mathcal{E}}{h} + \frac{h^2 \cdot M}{6} \geq \left| f'(x_i) - \frac{\tilde{f}_f(x_{i+1}) - \tilde{f}_f(x_{i-1})}{2h} \right|$$

$\therefore$ the optimum step, $\boxed{h_{opt} = \sqrt[3]{\frac{3\mathcal{E}}{M}}}$

-

+ Rule of thumb: when dividing, look for two nearly equal numbers and divide

example: chop to two decimal place: $x = 0.12 = z$, $y = 0.02$

then $\frac{x \cdot y}{z} = ?$

if $(0.12 \cdot 0.02) \cdot \frac{1}{0.12}$ is done:

$0.12 \cdot 0.02 = 0.0024$, stored as 0, error $= 100\%$

if $0.12 \cdot \left(\frac{0.02}{0.12}\right)$ is done, $\frac{0.02}{0.12} = 0.16$

$\rightarrow f_l = 0.12 \cdot 0.16 = 0.01 \rightarrow 50\%$ error

✓ if $\left(\frac{0.12}{0.12}\right) \cdot 0.02$ is done $\rightarrow 1 \cdot 0.2 = 0\%$ error

- Chopping or rounding is done after every arithmetic operation between two numbers in a computer.

Subject Suggested questions: Chapter4 Date 21/3/2021 No.

4.1: $\because \cos(\frac{\pi}{3}) = 0.5$

0) $e_t = -0.5$ $\%e_t = -100\%$

1) $1 - \frac{\pi^2}{18} = 0.45168864 \rightarrow e_t = 0.09662292 \rightarrow \%e_t = 9.662\%$

2) $1 - \frac{\pi^2}{18} + \frac{\pi^4}{1944} = 0.501796 \rightarrow e_t = -0.01796 \rightarrow \%e_t = -0.359\%$

4.4:

$$f(x_{i+1}) = f(x_i) + f'(x_i) \cdot h + \frac{f''(x_i) h^2}{2!} + \frac{f^{(3)}(x_i) h^3}{3!} + \frac{f^{(4)}(x_i) h^4}{4!}$$

0) $f(2.5) = 0 \rightarrow \%e_t = 100\%$

1) $f(2.5) = 0 + 1.5 \rightarrow \%e_t = -63.7\%$

2) $f(2.5) = 0 + 1.5 + \frac{-1 \cdot (1.5)^2}{2} \rightarrow \%e_t = 59.07\%$

3) $f(2.5) = 0 + 1.5 + \frac{-9}{8} + \frac{2 \cdot (1.5)^3}{6} \rightarrow \%e_t = -63.7\%$

4) $f(2.5) = 0 + 1.5 - \frac{9}{8} + \frac{9}{8} - \frac{6 \cdot (1.5)^4}{24} \rightarrow \%e_t = 74.4\%$

4.5: $\because f(3) = 554$

0) $f(3) = -62 \rightarrow \%e_t = 111.19\%$

1) $f(3) = -62 + 70 \cdot 2 \rightarrow \%e_t = 85.92\%$

2) $f(3) = -62 + 140 + \frac{138}{2} \cdot 2^2 \rightarrow \%e_t = 36.1\%$

3) $f(3) = -62 + 140 + 276 + \frac{150}{6} \cdot 2^3 \rightarrow \%e_t = 0\%$

4.6: $\because x = 2 \wedge h = 0.2$ $f(2) = 102$ $f'(2) = 283$

forward:

$$f'(2) = \frac{f(2.2) - f(2)}{0.2} = 312.8 \rightarrow \%e_t = -10.53\%$$

backward:

$$f'(2) = \frac{f(2) - f(1.8)}{0.2} = 255.2 \rightarrow \%e_t = 9.82\%$$

centered:

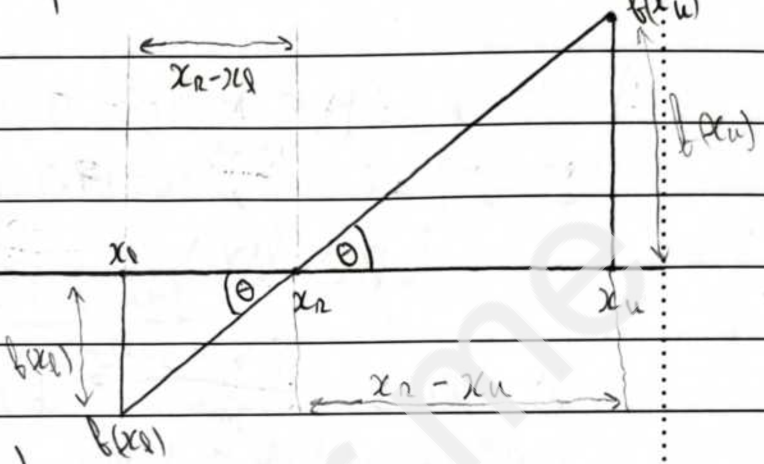
$$f'(2) = \frac{f(2.2) - f(1.8)}{0.4} = 284 \rightarrow \%e_t = -0.3533\%$$

Subject: false position method: Date 26/3/2021 No.

* false position method in linear interpolation

$$\because \tan(\theta) = \frac{f(x_u)}{x_r - x_u}$$

$$= \frac{f(x_l)}{x_r - x_l}$$



$$\Rightarrow \frac{f(x_u)}{x_r - x_u} = \frac{f(x_l)}{x_r - x_l}$$

$$\Rightarrow f(x_u)\cdot(x_r - x_l) = f(x_l)\cdot(x_r - x_u)$$

$$\Rightarrow \quad x_r = \frac{f(x_u)\cdot x_l}{f(x_u) - f(x_l)} - \frac{f(x_l)\cdot x_u}{f(x_u) - f(x_l)} + x_u - x_u$$

$$\therefore \boxed{x_r = x_u - \frac{f(x_u)\,[x_l - x_u]}{f(x_l) - f(x_u)}}$$

example from notes:

$\because f(x) = e^{-x} - x \quad \wedge \quad x_{l0} = 0 \ , \ x_{u0} = 1$

① $x_r = x_u - \dfrac{f(x_u)\cdot[x_l - x_u]}{f(x_l) - f(x_u)} = 0.612699$

$\therefore f(x_r) \Rightarrow$ negative value, hence x_r calculated $> x_r$ actual

② $x_u^{new} = 0.612699$

$$\Rightarrow x_r = 0.612699 - \frac{(-0.07081266)\cdot[-0.612699]}{0 - (-0.07081266)}$$

$$\Rightarrow x_r = 0.5721813 \Rightarrow f(x_r) = -7.881289 \times 10^{-3}$$

③ $x_u^{new} = 0.5721813$

$$\Rightarrow x_r = 0.56790$$

$\because x_r(\text{exact}) = 0.567143 \Rightarrow \boxed{\% AE_t = -0.09877\ \%}$

Subject Chapter 5: Suggested questions Date 29/3/2021 No.

5.1 : $\because f(x) = -0.6x^2 + 2.4x + 5.5 \rightarrow x = 5.62859$

or $x = -1.62859$

c) $\because x_u = 10 \wedge x_l = 5$

① $x_r^{(1)} = \frac{10+5}{2} = 7.5 \quad \%\varepsilon_t = -33.25\%$

$f(x_l)\cdot f(x_r) = \frac{-41}{4}\cdot 2.5 < 0 \rightarrow x_r^{true} < x_r = 7.5$

② $x_r^{(2)} = \frac{7.5+5}{2} = 6.25 \quad \%\varepsilon_t = -11.04\%$

$\rightarrow f(x_l)\cdot f(x_r) = -\frac{47}{16}\cdot 2.5 < 0 \rightarrow x_r^{true} < 6.25$

③ $x_r^{3} = \frac{6.25+5}{2} = 5.625 \rightarrow \%\varepsilon_t = 0.064\%$

$f(x_l)\cdot f(x_r) = \frac{1}{64}\cdot 2.5 > 0 \rightarrow x_r^{true} > 5.625$

- estimated percentage error can be calculated by:

$$\left|\frac{x_u - x_l}{x_u + x_l}\right| \times 100\%$$

5.4 : $x_r = -0.447 \wedge f(x) = -3x^3 + 19x^2 - 20x - 13$

b) ① $x_r^{(1)} = -0.5 \rightarrow \%\varepsilon_t = 11.86\%$

$\because f(x_l)\cdot f(x_r) = \frac{17}{8}\cdot 29 > 0 \rightarrow x_r^{true} > -0.5$

② $x_l = -0.5 \rightarrow x_r^{(2)} = -0.25 \rightarrow \%\varepsilon_t = 44.07\%$

$\rightarrow f(x_l)\cdot f(x_r) = \frac{17}{8}\cdot \frac{-433}{64} < 0 \rightarrow x_r^{true} < -0.25$

③ $x_r = -0.375 \rightarrow \%\varepsilon_t = 16.11\%$

$\rightarrow f(x_l)\cdot f(x_r) = \frac{17}{8}\cdot -2.6699 < 0 \rightarrow x_r^{true} < -0.375$

④ $x_r = -0.4375 \rightarrow \%\varepsilon_t = 2.12\%$

$f(x_l)\cdot f(x_r) = \frac{17}{8}\cdot -0.3621 < 0 \rightarrow x_r^{true} < -0.4375$

⑤ $x_r = -0.46875 \rightarrow \%\varepsilon_t = -4.866\%$

$f(x_l)\cdot f(x_r) = \frac{17}{8}\cdot 0.8588 > 0 \rightarrow x_r^{true} \geq -0.46875$

⑥ $x_u = -0.4375 \wedge x_l = -0.46875 \rightarrow x_r = -0.453125$

$\rightarrow \%\varepsilon_t = -1.37\%$

$f(x_l) \cdot f(x_r) = 0.8588 \cdot 0.242933 > 0 \rightarrow x_r^{true} > -0.453125$

⑦ $x_u = -0.4375 \wedge x_l = -0.453125 \rightarrow x_r = -0.4453125$

$\rightarrow \%\varepsilon_t = 0.377\%$

c) $$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

① $x_r = - \frac{-13 \cdot -1}{29 + 13} = \frac{-13}{42} \rightarrow \%\varepsilon_t = 30.95\%$

$\therefore f(x_r) \cdot f(x_l) = 29 \cdot -4.9 < 0 \rightarrow x_r^{true} < -\frac{13}{42}$

② $f(x_u) = -4.9 \rightarrow x_r = -0.40933$

$\rightarrow \%\varepsilon_t = 8.43\%$

$\therefore f(x_r) \cdot f(x_l) = 29 \cdot -1.4242 < 0 \rightarrow x_r^{true} < -0.40933$

③ $f(x_u) = -1.4242 \rightarrow x_r = -0.43697 \rightarrow \%\varepsilon_t = 2.259\%$

$\therefore f(x_r) \cdot f(x_l) = 29 \cdot -0.3824 \rightarrow x_r^{true} < -0.43697$

④ $f(x_u) = -0.382398 \rightarrow x_r = -0.444297$

$\rightarrow \%\varepsilon_t = 0.6047\%$

5.5: $\sin(x) = x^3 \rightarrow \sin(x) - x^3 = 0$ $\qquad x_r^{true} = 0.93$

① $x_r = 0.75 \rightarrow f(x_l) \cdot f(x_r) = 0.2497 \cdot 0.3544 > 0 \rightarrow x_r^{true} > 0.75$

② $x_l = 0.75 \rightarrow x_r = 0.875$

$f(x_l) \cdot f(x_r) = 0.2597 \cdot 0.976216 > 0 \rightarrow x_r^{true} > 0.875$

③ $x_l = 0.875 \rightarrow x_r = 0.9375 \rightarrow \%\varepsilon_t = -0.806\%$

$\%\varepsilon_a = \left|\frac{x_r - x_l}{x_r + x_l}\right| \times 100 = 6.66\%$

$\therefore f(x_l) \cdot f(x_r) = 0.976216 \cdot -0.01789 < 0 \rightarrow x_r^{true} < 0.9375$

④ $x_u = 0.9375 \rightarrow x_r = 0.90625 \rightarrow \%\varepsilon_a = 3.448\%$

$\therefore f(x_l) \cdot f(x_r) = 0.97216 \cdot 0.0429 > 0 \rightarrow x_r^{true} > 0.90625$

⑤ $x_l = 0.90625 \rightarrow \boxed{x_r = 0.921875} \wedge \%\varepsilon_a = 1.695\%$

5.6: $\ln(x) - \frac{0.7}{4} = 0$ $x_r^{true} = 1.191$

b) ① $x_r = 1.25$, $f(x_l) \cdot f(x_r) = 0.0481435 \cdot -0.868149 < 0$

$\rightarrow x_r^{true} < 1.25$

② $x_r = 0.875$, $f(x_l) \cdot f(x_r) = -0.868147 \cdot -0.30853 > 0$

$\rightarrow x_r^{true} > 0.875$

③ $\boxed{x_r = 1.0625}$ $\%\varepsilon_t = 10.59\%$

c) $x_r = x_u - \frac{f(x_u)[x_l - x_u]}{f(x_l) - f(x_u)}$

① $x_r = 1.43935$, $f(x_l) \cdot f(x_r) = -0.868147 \cdot 0.18919$

$\rightarrow x_r < 1.43935$

② $x_r = 1.27127$, $f(x_l) \cdot f(x_r) = -0.868147 \cdot 0.0660 < 0$

$\rightarrow x_r < 1.27127$

③ $\boxed{x_r = 1.21753}$, $\%\varepsilon_t = -2.2295\ \%$

$\%\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100 = 4.4138\ \%$

Subject Chapter 6: Open methods Date 29/3/2021 No.

6.1:

\+ given a function, $f(x)$, it must be rearranged to have x at one side and $g(x)$ at the other

\- set $f(x) = 0$, manipulate to form $x = g(x)$

e.g: $x^2 - 2x + 3 = 0 \rightarrow x = \frac{x^2+3}{2}$

e.g: $\sin(x) = 0 \rightarrow \sin(x) + x - x = 0 \rightarrow x = \sin(x) + x$

example: $f(x) = e^{-x} - x$

① set $f(x) = 0 \rightarrow e^{-x} - x = 0 \rightarrow x = e^{-x}$

$$x_{i+1} = e^{-x_i}$$

② iterate: $x_0 = 0 \rightarrow x_1 = e^{-0} = 1$

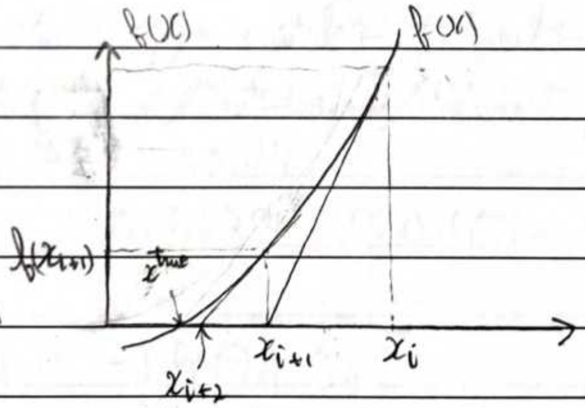
$x_1 = 1 \rightarrow x_2 = e^{-1} = 0.36787$

$x_2 = 0.36787 \rightarrow x_3 = e^{-0.36787} = 0.6922$

$x_3 = 0.6922.. \rightarrow x_4 = e^{-0.6922} = 0.50047..$

6.2: The newton - Raphson method

\+ since $f'(x_i)$ represents the slope at point x_i, and x^{true} represents the true root of $f(x)$. hence, x_{i+1} is an estimate of the root.



$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

- steps to use newton-Raphson method:

1- evaluate $f'(x)$ symbolically

2- Use an initial guess of the root (x_i) to start iterating

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3- find the relative approximate error and compare to required:

$$\% e_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100 \quad \text{or} \quad \left| \frac{x_{new} - x_{old}}{x_{new}} \right| \times 100$$

example: $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$

$\rightarrow f'(x) = 3x^2 - 0.33x + 0$, $x_0 = 0.05$

① $x_1 = 0.05 - \frac{f(0.05)}{f'(0.05)} = 0.0624222$, $\% e_a = 19.9\%$

no SF correct

② $x_2 = 0.062422 - \frac{f(0.062422)}{f'(0.062422)} = 0.0623775$, $\% e_a = -0.07121\%$

at least 2 SF correct

③ $x_3 = 0.0623776$, $\% e_a \approx 0\%$, at least 4 SF correct

- an absolute relative approximate error of 5% or less for one significant figure to be correct

Subject Open methods: continued Date 11/4/2021 No.

* The secant method:

- Similar to the newton raphson method, except the derivative is evaluated by a backward finite difference:

$$\boxed{f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}} \quad ①$$

$$\therefore \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \quad ②$$

Sub ① in ②:

$$x_{i+1} = x_i - \frac{(x_{i-1} - x_i)(f(x_i))}{f(x_{i-1}) - f(x_i)}$$

example 6.6: $f(x) = e^{-x} - x$, $x_{-1} = 0$, $x_0 = 1$

① $\rightarrow f'(x_0) = \frac{f(0) - f(1)}{0 - 1} = -1.63212059$

$\rightarrow x_{i+1} = 1 - \frac{e^{-1} - 1}{-1.63212059} = \boxed{0.6126998368}$

$\therefore$ true root $= 0.56714329$, $\%\varepsilon_t = 8\%$

② $x_0 = 1 \rightarrow f'(x_1) = -1.44928$

$x_1 = 0.61270 \wedge x_2 = 0.563838 \approx 0.56384$

$\rightarrow \%\varepsilon_t = 0.58\%$

③ $x_1 = 0.61270 \rightarrow f'(x_2) = -1.55534234$

$x_2 = 0.56384 \wedge x_3 = 0.5671703453$

$\rightarrow \%\varepsilon_t = 4.77 \times 10^{-3}\ \%$

- the two values used in estimating the root can both lie on its same side (e.g. both larger)

- modified secant method: using a small perturbation factor (δ)

$$\rightarrow \quad f'(x_i) \cong \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

$$\wedge \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\rightarrow \boxed{x_{i+1} = x_i - \frac{\delta x_i \cdot f(x_i)}{f(x_i + \delta x_i) - f(x_i)}}$$

example 6.8: $\delta = 0.01$, $f(x) = e^{-x} - x$, $x_0 = 1$

① $f'(x_i) = \dfrac{f(1 + 0.01) - f(1)}{0.01} = -1.36604616$

$\rightarrow x_1 = 0.537262665 \rightarrow \% \varepsilon_t = 5.3\%$

② $f'(x_1) = -1.58278 \rightarrow x_2 = 0.56701 \rightarrow \% \varepsilon_t = 0.0236\%$

③ $x_3 = 0.567143 \rightarrow \% \varepsilon_t = -2.365 \times 10^{-5} \%$

example from slides: $f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$

$x_{-1} = 0.02$, $x_0 = 0.05$

$x_1 = 0.05 - \dfrac{f(0.05)}{f'(0.05)}$, $f'(0.05) = \dfrac{f(0.02) - f(0.05)}{0.02 - 0.05}$

① $\rightarrow x_1 = 0.0646143991 \approx 0.0646$

$\therefore \% \varepsilon_a = \left| \dfrac{x_{new} - x_{old}}{x_{new}} \right| \times 100\% = 22.6126\%$

② $x_0 = 0.05$ $\rightarrow$ $x_2 = 0.06241$

$x_1 = 0.0646$ $\quad \% \varepsilon_a = 3.525\%$

③ $x_3 = 0.062379 \rightarrow \% \varepsilon_a = -0.0523\%$ 5 SF correct

0.5% or less gives 5 correct SF

6.5: multiple roots:

- a multiple root results from a point that is tangent to the x-axis
- odd multiple roots generally cross the x-axis while even multiples are just tangent to it at the root point

  example: $f(x) = (x-3)(x-1)(x-1)$ is tangent to x-axis at $x=1$

  $f(x) = (x-3)(x-1)(x-1)(x-1)$ crosses the x-axis while tangent to it all at $x=1$
- at even multiple roots, the bracketing methods cannot be used as the function does not change sign at that particular root. hence, open methods must be used.
- another problem is that a function that is tangent to the x-axis at a given point has a zero gradient at that point. Thus the newton-Raphson and secant methods cannot be used since the derivative is zero at that point (cannot divide by zero)
- since the newton-Raphson and secant methods become linearly convergent for multiple roots (rather than quadratically convergent for single roots) a modification to the formula is used:

$$x_{i+1} = x_i - m \cdot \frac{f(x_i)}{f'(x_i)}$$

m = multiplicity of root (e.g. m = 2 for double root)

this modification relies on prior knowledge of root multiplicity
- another modification defines a new function from $f(x)$ this new function has the same roots as $f(x)$

$$u(x) = f(x) / f'(x)$$

, sub this function in the equation for x_{i+1}

$$\therefore x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\rightarrow \text{modified: } \boxed{x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}}$$

$$\boxed{u'(x_i) = \frac{f'(x_i) \cdot f'(x_i) - f''(x_i) \cdot f(x_i)}{[f'(x_i)]^2}}$$

$$\therefore \frac{u(x_i)}{u'(x_i)} = \frac{f(x_i)\cdot[f'(x_i)]^2}{f'(x_i)\left[[f'(x_i)]^2 - f''(x_i)\cdot f(x_i)\right]} = \frac{f(x_i)\cdot f'(x_i)}{[f'(x_i)]^2 - f(x_i)\,f''(x_i)}$$

$$\boxed{\therefore\ x_{i+1} = x_i - \frac{f(x_i)\cdot f'(x_i)}{[f'(x_i)]^2 - f(x_i)\cdot f''(x_i)}}$$

example 6.10: $f(x) = x^3 - 5x^2 + 7x - 3$, $x_0 = 0$

a) standard newton raphson: $f'(x) = 3x^2 - 10x + 7$

$$x_{i+1} = x_i - \frac{x_i^3 - 5x_i^2 + 7x_i - 3}{3x_i^2 - 10x_i + 7} \qquad \text{true root} = 1$$

① $x_1 = \frac{3}{7} \to \%\varepsilon_t = 57\%$ | ② $x_2 = 0.6857143 \to \%\varepsilon_t = 31.4\%$

③ $x_3 = 0.8328654 \to \%\varepsilon_t = 16.7\%$ | ④ $x_4 = 0.9133299 \to \%\varepsilon_t = 8.7\%$

⑤ $x_5 = 0.9558 \to \%\varepsilon_t = 4.42\%$ | ⑥ $x_6 = 0.9776551 \to \%\varepsilon_t = 2.2\%$

- if we use the first modified newton raphson method

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

we get ~ 0.5% accuracy after two iterations

b) modified newton-raphson:

$$x_{i+1} = x_i - \frac{(x_i^3 - 5x_i^2 + 7x_i - 3)(3x_i^2 - 10x_i + 7)}{[3x_i^2 - 10x_i + 7]^2 - (x_i^3 - 5x_i^2 + 7x_i - 3)\cdot(6x_i - 10)}$$

① $x_1 = 1.105263 \to \%\varepsilon_t = 10.53\%$

② $x_2 = 1.00308166 4 \to \%\varepsilon_t = 0.3082\%$

③ $x_3 = 1.000002381$

hence the second method converges quadratically

- the single root $x = 3$ can be found by making an initial guess of $\boxed{x_0 = 4}$

6.6: systems of nonlinear equations:

- solution to simultaneous equations is a set of x values that result in all equations equalling zero simultaneously (hence the name)

example 6.11: $u(x,y) = x^2 + xy - 10 = 0$ — ①

$V(x,y) = y + 3xy^2 - 57 = 0$ — ②

must find values of x and y where $V(x,y)$ and $u(x,y)$ are zero

① $x^2 + xy - 10 = 0 \rightarrow \cancel{x}(1+y)+x^2-10=x$

$\rightarrow xy = 10 - x^2 \rightarrow \boxed{x_{i+1} = \frac{10-x_i^2}{y_i}}$ ✗

at $x_0 = 1.5$

② $y_{i+1} = 57 - 3x_i y_i^2$ $y_0 = 3.5$ ✗

$i_1: x_1 = \frac{10-(1.5)^2}{3.5} = 2.2143$ | $y_1 = -24.3755$ ✗ diverges

① $x^2 = -xy + 10 \rightarrow x_{i+1} = \sqrt{10 - x_i y_i}$

$x_0 = 1.5$

② $3xy^2 = 57 - y \rightarrow y_{i+1} = \sqrt{\frac{57-y_i}{3x_i}}$

$y_0 = 3.5$

$i_1 = x_1 = 2.17945$, $y_1 = 2.86051$

$i_2 = x_2 = 1.94053$, $y_2 = 3.04955$

$\therefore$ x and y are approaching their true values of 2 & 3

- fixed point iteration (as done in the previous example) depends on the formulation of the initial equations for x and y and the initial guesses, specific formulas might diverge and guesses too far from the true value might cause divergence even when using the right formulas

- Criteria to guarantee convergence when using fixed point iteration:

$$\boxed{\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| < 1 \quad \lambda \quad \left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| < 1}$$

The above conditions are restrictive and fixed point iteration should be avoided

* Newton-Raphson for solving nonlinear simultaneous equations:

$$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{|J_{u,v}|}$$

$$y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{|J_{u,v}|}$$

Where the denominator of the two above equations is the determinant of the jacobian of the system

$$|J_{u,v}| = \frac{\partial u_i}{\partial x} \cdot \frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y} \cdot \frac{\partial v_i}{\partial x}$$

example 6.11:

$$u(x,y) = x^2 + xy - 10 = 0 \qquad x_0 = 1.5$$

$$v(x,y) = y + 3xy^2 - 57 = 0 \qquad y_0 = 3.5$$

$$\frac{\partial u_0}{\partial x} = 2x + y \Big|_{x=1.5,\, y=3.5} = 6.5$$

$$\lambda \ \frac{\partial v_0}{\partial x} = 3y^2 \Big|_{y=3.5} = 36.75$$

& $\frac{\partial u_0}{\partial y} = x|_{x=1.5} = 1.5$ & $\frac{\partial v_0}{\partial y} = 1+6xy|_{x=1.5,\, y=3.5} = 32.5$

& $u_0 = -2.5$, $v_0 = 1.625$

$\rightarrow x_1 = 1.5 - \frac{(-2.5)(32.5) - (1.625)(1.5)}{(6.5)(32.5) - (1.5)(36.75)} = 1.5 - \frac{-83.6875}{156.125} = 2.03603$

$\rightarrow y_1 = 3.5 - \frac{(1.625)(6.5) - (-2.5)(36.75)}{156.125} = 3.5 - \frac{102.4375}{156.125} = 2.8438751$

- the results are converging to the true values of x & y
- the newton raphson approach will often diverge if the initial guesses aren't close enough to the true values

example from notes:

$u(x,y) = x^2 + y^2 - 4 = 0$

$v(x,y) = (x-a)^2 + (y-b)^2 - 4 = 0 \rightarrow x^2 - 2ax - a^2 + y^2 - 2by - b^2 - 4 = 0$

* derivation:

$\rightarrow \overbrace{u(x_{i+1}, y_{i+1})}^{u_{i+1}} = \overbrace{u(x_i, y_i)}^{u_i} + (x_{i+1} - x_i)\frac{\partial u_i}{\partial x} + (y_{i+1} - y_i)\frac{\partial u_i}{\partial y} = 0$

& $v_{i+1} = v_i + \overbrace{(x_{i+1} - x_i)}^{\Delta x}\frac{\partial v_i}{\partial x} + \overbrace{(y_{i+1} - y_i)}^{\Delta y} \cdot \frac{\partial v_i}{\partial y} = 0$

$$\therefore \boxed{\begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -u_i \\ -v_i \end{bmatrix}}$$

$\because v_{i+1} = u_{i+1} = 0 \rightarrow -u_i = \Delta x \frac{\partial u_i}{\partial x} + \Delta y \frac{\partial u_i}{\partial y}$

& $-v_i = \Delta x \frac{\partial v_i}{\partial x} + \Delta y \frac{\partial v_i}{\partial y}$

example from notes :

$$U(x,y) = x^2 + xy - 10 = 0 \quad — ① \qquad x_0 = 1.5$$

$$V(x,y) = y + 3xy^2 - 57 = 0 \quad — ② \qquad y_0 = 3.5$$

$$\begin{bmatrix} \frac{\partial U_i}{\partial x} & \frac{\partial U_i}{\partial y} \\ \frac{\partial V_i}{\partial x} & \frac{\partial V_i}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -U_i \\ -V_i \end{bmatrix}$$

$$\frac{\partial U_0}{\partial x} = 2x + y \Big|_{x=1.5,\, y=3.5} = 6.5 \quad \Big| \quad \frac{\partial U_0}{\partial y} = 1.5$$

$$\frac{\partial V_0}{\partial x} = 36.75 \quad \Big| \quad \frac{\partial V_0}{\partial y} = 32.5 \quad \Big| \quad U_0 = -2.5 \quad \Big| \quad V_0 = 1.625$$

$$\rightarrow \begin{bmatrix} 6.5 & 1.5 \\ 36.75 & 32.5 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.625 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.5360 \\ -0.6561 \end{bmatrix} \rightarrow \begin{matrix} x = 2.036 \\ y = 2.8439 \end{matrix}$$

- Root multiplicity :

def: ① if (c) is a root of a function $f(x)$. Then the root has multiplicity (m) if $\frac{f(x)}{(x-c)^m}$ has a limit at c that converges to a nonzero number

example : find m of root 0 for $f(x) = 1 - \cos(x)$

$$\rightarrow m=1 : \quad \frac{1-\cos x}{(x-0)^1} = \frac{0}{0} \text{ for } \lim_{x\to 0}$$

apply L'hospital : $\frac{\frac{\partial}{\partial x}[1-\cos(x)]}{\frac{\partial}{\partial x}[(x-0)^1]} = \frac{\sin(x)}{1} = 0$

$$\therefore m \neq 1$$

→ $m=2$: $\lim_{x\to0}\left[\frac{1-\cos x}{x^2}\right]$ apply L'hopital : $\lim_{x\to0}\left[\frac{\sin x}{2x}\right]$

apply L'hopital again : $\lim_{x\to0}\left[\frac{\cos x}{2}\right] = \frac{1}{2}$ $\boxed{\text{converge}}$

hence $\boxed{m=2}$

def: ② if c is a root of a function, then Root(c) has multiplicity (m) where: $f(c)=0$, $f'(c)=0$, --- $f^{(m-1)}(c)=0$ but $f^{(m)}(c)\neq0$

example: $f(x) = 1-\cos(x)$, $c=0$.

$f(c) = 0$, $f'(c)=0$, $f''(c) = -1 \neq 0$

$\therefore$ $m=2$ since two derivatives were taken.

Subject gauss-seidel method: (11.2) and jacobi Date 12/4/2021 No.

- assume $[A]\{x\} = \{B\}$, contains n equations

if $n = 3$ → A is a 3×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

→ $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

→ $a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

→ $a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

$$\therefore \boxed{x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}}$$

$$\& \quad \boxed{x_2 = \frac{b_2 - a_{21} x_1 - a_{23} x_3}{a_{22}}}$$

$$\& \quad \boxed{x_3 = \frac{b_3 - a_{31} x_1 - a_{32} x_2}{a_{33}}}$$

- an initial guess for two xs is required for the gauss-seidel method since the most recent value for x_i is used (a third value is required for error calculations)
- Jacobi method uses the initial guesses in every equation for the first iteration then updates
- initial guess of zero can be made for all xs
- approximate error: $|\varepsilon_{a,i}| = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100\%$

* Condition for convergence: the absolute value of the diagonal x must be larger than the sum of the absolute values of the other elements in its row.

$$\boxed{|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|}$$

- if condition for convergence is not satisfied then rows must be exchanged:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

if $|a_{22}| \leq |a_{21}| + |a_{23}|$ then condition is not satisfied

assume $|a_{32}| > |a_{33}| + |a_{31}|$ & $|a_{23}| > |a_{21}| + |a_{22}|$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_3 \\ b_2 \end{bmatrix}$$

hence row a_{3x} was exchanged with a_{2x} and b_2 with b_3

example 11.3:

check condition: $3 > 0.1 + 0.2$ | $7 > 0.1 + 0.3$ | $10 > 0.3 + 0.2$ satisfied

$$\rightarrow x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \quad \Big| \quad x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}, \quad x_1^0 = 0 = x_2^0 = x_3^0$$

① gauss seidel:

$$x_1^{\boxed{1}} = \frac{7.85}{3} = \boxed{2.61667}, \quad x_2^{\boxed{1}} = \frac{-19.3 - \boxed{0.1(2.61667)} + 0.3(0)}{7} = \boxed{-2.794524}$$

$$x_3^{\boxed{1}} = \frac{71.4 - \boxed{0.3(2.61667)} + \boxed{0.2(-2.794)}}{10} = 7.005610$$

$$\therefore x_1^{\boxed{2}} = \frac{7.85 + 0.1(+2.794524) + 0.2(7.00561)}{3} = \boxed{2.990557}$$

$$x_2^{\boxed{2}} = \frac{-19.3 - 0.1\boxed{(2.990557)} + 0.3(7.00561)}{7} = \boxed{-2.499625}$$

$$x_3^{\boxed{2}} = \frac{71.4 - 0.3(2.990557) + 0.2\boxed{(-2.499625)}}{10} = 7.000291$$

$$\therefore |\varepsilon_{a,1}| = \left|\frac{2.990557 - 2.616667}{2.990557}\right| \times 100 = 12.5\%$$

② jacobi: $x_1^0 = \boxed{0} = x_2^0 = x_3^0$

$$\rightarrow x_1^{①} = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = \boxed{2.616667},\quad x_2^{①} = \frac{-19.3 - 0.1(0) + 0.3(0)}{7} = \boxed{-2.757143}$$

$$x_3^{①} = \frac{71.4 - 0.3\boxed{(0)} + 0.2\boxed{(0)}}{10} = \boxed{7.14}$$

$$x_1^{\boxed{2}} = \frac{7.85 + 0.1\boxed{(-2.757143)} + 0.2(7.14)}{3} = 3.0007619$$

$$x_2^{②} = \frac{-19.3 - 0.1\boxed{(2.616667)} + 0.3\boxed{(7.14)}}{7} = -3.0257619$$

$$x_3^{\boxed{2}} = \frac{71.4 - 0.3(2.616667) + 0.2(-2.757143)}{10} = 7.006357$$

- the jacobi iteration method converges more slowly than gauss-seidel but can be useful for some equations

Subject mid exam practice Date 15/4/2021 No.

Chapter 6 suggested questions:

6.1: ∵ $f(x) = 2\sin(\sqrt{x}) - x \rightarrow x = 2\sin(\sqrt{x})$

① → $x_{i+1} = 2\cdot\sin(\sqrt{0.5}) = 1.2993$

$\%\varepsilon_a = 61.5\%$

② $x_2 = 2\cdot\sin(\sqrt{0.5}) = 1.8171477$ $\%\varepsilon_a = 28.5\%$

③ $x_3 = 1.950574$ $\%\varepsilon_a = 6.84\%$

④ $x_4 = 1.96974$ $\%\varepsilon_a = 0.973\%$

⑤ $x_5 = 1.9720688$ $\%\varepsilon_a = 0.118\%$

6.3: ∵ $f(x) = -x^2 + 1.8x + 2.5 \rightarrow x = -x^2 + 2.8x + 2.5$

a) ① -8.5 ① 3.2416 or $x = \sqrt{1.8x + 2.5}$

② -93.55 ② 2.93327 $\%\varepsilon_a =$

③ -9011.0425 ③ 2.489246

④ 2.76238

⑤ 2.726955

⑥ 2.72186 $\%\varepsilon_a = -0.18716\%$

⑦ 2.72017

⑧ $\boxed{2.719616}$ $\%\varepsilon_a = -0.02\%$

⑨ 2.7194318 $\%\varepsilon_a = -6.793\%$

m

b) ∵ $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{-x_i^2 + 1.8x_i + 2.5}{-2x_i + 1.8}$

① 3.34366

② 2.80133

③ 2.72111

④ 2.71934

⑤ 2.71930 $\%\varepsilon_a = 0\%$

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6.5: $x_{i+1} = x_i - \frac{-2+6x_i-4x_i^2+0.5x_i^3}{6-8x_i+1.5x_i^2}$

a) ① -4.849123

② -2.573919

③ -1.1376259

④ -0.272738

⑤ 0.2028299

⑥ 0.4153539

⑧ 0.47459 since the function has a double

⑨ $\boxed{0.47459}$ root → tangential at $x = 0.474659$

b) ① ~~-4.849123~~ -3939

② ~~-2.73419~~ -2625

③ ~~-1.137626~~ -1749

④ ~~-0.27724~~

⑤ ~~0.2028~~

⑧ 0.474652

⑨ 0.474577

⑩ 0.474572

6.7: $f(x) = \sin(x) + \cos(1+x^2) - 1$

$\wedge \quad f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$

a) $f'(x_i) = \frac{-0.574675 + 1.69795}{-2} = -0.561637$

$\wedge \quad f(x_i) = -1.69795$

$$x_{i+1} = x_i - \frac{f(x_i)\cdot(x_{i-1}-x_i)}{f(x_{i-1}) - f(x_i)}$$

① $f(x_{i-1}) = -0.574675$ ^ $f(x_i) = -1.6795$

→ $x_1 = -0.02321$ $f(x_1) = -0.48336$

② $x_{i-1} = 3$, $x_i = -0.02321$

→ $x_2 = -1.22635$ $f(x_2) = -2.74475$

③ $x_2 = -1.22635$, $x_1 = -0.02321$

→ $x_3 = 0.233455$, $f(x_3) = -0.274715$

④ $x_3 = 0.233455$, $x_2 = -1.22635$

→ $x_4 = 0.396369$

---

b) ① $f(x_{-1}) = -0.99663$, $f(x_0) = 0.166396$

$x_{-1} = 1.5$, $x_0 = 2.5$

→ $x_1 = 2.35693$ → $f(x_1) = 0.669843$

② $f(x_0)$, ~~$f(x_1)$~~ , x_0 , x_1

→ $x_2 = 2.5493$ → $f(x_2) = -0.0828725$

③ $f(x_2)$, $f(x_1)$, x_2 , x_1

→ $x_3 = 2.52044$, $f(x_3) = 0.062595$

④ $x_4 = 2.532$, $f(x_4) = 1.13769 \times 10^{-3}$.

Subject Date No.

6.14: $f(x) = x^3 - 2x^2 - 4x + 8$, $x_0 = 1.2$

a) $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $f'(x) = 3x^2 - 4x - 4$

① $x_1 = 1.65714$ $f''(x) = 6x - 4$

② $x_2 = 1.837002$

③ $x_3 = 1.9202698$

④ $x_4 = 1.96054$

⑤ $x_5 = 1.980391$

b) knowing that $m = 2$ → modified: $x_{i+1} = x_i - m \frac{f(x_i)}{f'(x_i)}$

→ ① $x_1 = 2.1143$

② $x_2 = 2.00197$

③ $x_3 = 2.00000031$

c) not knowing multiplicity, $x_{i+1} = x_i - \frac{f(x_i) f'(x_i)}{[f'(x_i)]^2 - f(x_i) f''(x_i)}$

$$\rightarrow x_{i+1} = x_i - \frac{(x^3 - 2x^2 - 4x + 8)(3x^2 - 4x - 4)}{(3x^2 - 4x - 4)^2 - (6x - 4)(x^3 - 2x^2 - 4x + 8)}$$

① $x_1 = 1.898989$

② $x_2 = 1.99805$

③ $x_3 = 1.9999995$

④ $x_4 = x_3$

6.16: $x = 1.8$, $y = 3.6$ $\wedge$ $x = 3.6$, $y = 1.8$

$$\therefore x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x} \cdot \frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y} \frac{\partial v_i}{\partial x}}$$

$$y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x_i} - \frac{\partial v_i}{\partial x_i} \cdot u_i}{\frac{\partial u_i}{\partial x}}$$

$$\overset{2x-8}{\frac{\partial u_i}{\partial x}} \cdot \overset{2y}{\frac{\partial v_i}{\partial y}} - \overset{2y-8}{\frac{\partial u_i}{\partial y}} \cdot \overset{2x}{\frac{\partial v_i}{\partial x}} = 4xy - 16y - 4xy + 16x = \boxed{16x - 16y}$$

① : $16x - 16y = -28.8$, $u_i = 0$, $v_i = 0.2$, $\frac{\partial u_i}{\partial y} = -0.8$

$\rightarrow x_{i+1} = 1.8 - \frac{-0.2 \cdot -0.8}{-28.8} = 1.805556$

$\rightarrow y_{i+1} = 3.6 - \frac{-4.4 \cdot 0.2}{-28.8} = 3.569444$

② $16x - 16y = 28.8$, $v_i = 0.2$, $u_i = 0$, $\frac{\partial v_i}{\partial x} = 7.2$, $\frac{\partial u_i}{\partial x} = -0.8$

$\therefore x_{i+1} = 3.569444$

$\wedge \ y_{i+1} = 1.80555$

$f(x) = \frac{1}{x-1}$, $x_0 = 1.3$, $h = 0.3$

a) $f(x_{i+1}) = f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2!} h^2 + \boxed{R_2}$

$\rightarrow f(x_{i+1}) \approx \frac{1}{0.3} + 0.3 \cdot \frac{-1}{(0.3)^2} + \frac{2\,(0.3)^2}{2 \cdot (0.3)^3} \approx 3.33333$

b) $\because f_t(1.6) = 1.666667$ $R_n = f_{true} - f_{approx.}$

$\rightarrow R_2 = 1.666667 - 3.333333 = \boxed{-1.66667}$

c) $\% RE_t = \left| \frac{R_2}{f_t(1.6)} \right| \times 100\% \approx 100\%$

d) $\because R_2 = \frac{f^{(3)}(\xi)}{3!} (0.3)^3 \rightarrow -1.6666667 = -\frac{6}{(\xi-1)^4} \cdot \frac{(0.3)^3}{6}$

$\rightarrow (\xi - 1)^4 = 0.0162$

$\rightarrow \boxed{\xi = 1.35676}$

past mid :

Q1 : normalised floating point : $N = \mp M b^{\mp e}$, $\frac{1}{b} \le M < 1$

$M \geq 0.5$

sign | e mag | m mag

$\boxed{x_7 \; x_6} \; \boxed{x_5 \; x_4} \; \boxed{x_3 \; x_2 \; x_1}$

$x = 2.0777$, $y = 0.1891$

0	1	1	1	1	0	0	0.0625
0	1	1	1	1	0	1	
0	1	1	1	1	1	0	
0	1	1	1	1	1	1	0.109375
0	1	1	0	1	0	0	0.125
0	1	1	0	1	0	1	0.15625
0	1	1	0	1	1	0	0.1875
0	1	1	0	1	1	1	0.21875
0	0	1	0	1	0	0	2
0	0	1	0	1	0	1	2.5

$\therefore x \times y = 2.0777 \times 0.1891$

$= 2 \times 0.1875$

$= \boxed{0.375}$

b) $\% RE_t = 4.646 \%$

$x_0 = 3.89$

Q2) $f(x) = \ln(x) - x^2$

a) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$

& $h = 0.1$, $f'(x) = \frac{1}{x} - 2x \Rightarrow f''(x) = \frac{-1}{x^2} - 2$

$\Rightarrow f(3.99) = -13.7739 + (-0.75229) + (-0.0103304)$

$\therefore = \boxed{-14.5363304}$

b) $\varepsilon = 0.0005026$

assuming 2nd order approximation, $R_2 = \frac{f^{3}(\xi)}{3!}h^3$

& $\xi \in [1,2]$, $f^{(3)}(\xi) = \frac{2}{x^3} = \frac{2}{\xi^3}$

R_n is the absolute error for nth order approximation

$\therefore \frac{1}{3\xi^3} \cdot h^3 < 0.0005026$

if $\xi = 2 \Rightarrow h = 0.229339$

if $\xi = 1 \Rightarrow \boxed{h = 0.114669}$

Q3) $\boxed{n = 13}$

$\therefore E_a^n = \frac{\Delta x^0}{2^n} = \varepsilon = 0.0005026$

$\Rightarrow \Delta x^0 = 2^{13} \cdot 0.0005026 = 4.1173$

Q1) $x_0 = 1.08$, $f(x) = 0.5x + 3x^{1/2}$

a) $h = 0.5$

$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$

$f'(x_i) = \frac{3}{2\sqrt{x_i}} + \frac{1}{2}$, $f''(x_i) = -\frac{3}{4x^{3/2}}$

$\Rightarrow f(x_{i+1}) = 3.6577 + 0.97169 + (-0.08353)$

$= 4.54586$

b) $R = f_{true} - f_{approx} = 4.56094 - 4.54586$

$= \boxed{0.01508}$

Q2) a) by using taylor series approximations for two variables

b) $$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x} \cdot \frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y} \cdot \frac{\partial v_i}{\partial x}}$$

$x_0 = 4.25$, $y_0 = 1$, $u_0 = 13.5625$, $\frac{\partial v_0}{\partial y} = 1$

$\frac{\partial u_0}{\partial x} = 8.5$, $\frac{\partial v_0}{\partial x} = -0.04279$ | $v_0 = -0.95921$, $\frac{\partial u_0}{\partial y} = 0.5$

$$\rightarrow x_1 = 4.25 - \frac{13.5625 \cdot 1 - (-0.95921) \cdot 0.5}{(8.5)(1) - (0.5)(-0.04279)}$$

$$= 2.60225$$

or using matrices:

$$\begin{bmatrix} \frac{\partial u_i}{\partial x} & \frac{\partial u_i}{\partial y} \\ \frac{\partial v_i}{\partial x} & \frac{\partial v_i}{\partial y} \end{bmatrix} \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix} = \begin{bmatrix} -u_i \\ -v_i \end{bmatrix}$$

$$\begin{bmatrix} 8.5 & 0.5 \\ -0.04279 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_i \\ \Delta y_i \end{bmatrix} = \begin{bmatrix} -13.5625 \\ +0.95921 \end{bmatrix}$$

$\rightarrow \Delta x_0 = -1.6477$ $\wedge$ $\Delta y_0 = 0.8867$

$\therefore \boxed{x_1 = 2.6023}$ $\wedge$ $\boxed{y_1 = 1.8867}$

$\rightarrow \frac{\partial u_1}{\partial x} = 5.2046$, $\frac{\partial v_1}{\partial x} = -0.22231$ | $u_1 = 2.7153$

$\wedge$ $\frac{\partial u_1}{\partial y} = 0.5$, $\frac{\partial v_1}{\partial y} = 1$ | $v_1 =$

$$\rightarrow \begin{bmatrix} 5.2046 & 0.5 \\ -0.22231 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = \begin{bmatrix} -2.7153 \\ 0.10901 \end{bmatrix}$$

$\therefore \Delta x_1 = -0.5005 \rightarrow \boxed{x_2 = 2.1018}$

6.17: $\because \frac{\partial u_i}{\partial x} = -2x$, $\frac{\partial v_i}{\partial x} = 2\sin x$

$\frac{\partial u}{\partial y} = 1$ $\frac{\partial v_i}{\partial y} = 1$

$u = y-(x^2+1)$

$v = y-2\cos x$ $\therefore$ $\begin{bmatrix} -2x_i & 1 \\ 2\sin(x_i) & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -u_i \\ -v_i \end{bmatrix}$

① 1) $x_0 = 0.7$, $y_0 = 1.5$

$$\begin{bmatrix} -1.4 & 1 \\ 1.2884 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.01 \\ +0.02968 \end{bmatrix}$$

$\rightarrow \Delta x = 0.0148$, $\Delta y = 0.0107$

$\rightarrow x_1 = 0.7148$, $y_1 = 1.5107$

2) $\begin{bmatrix} -1.4296 & 1 \\ 1.3109 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 2.3904\times10^{-4} \\ -2.51301\times10^{-4} \end{bmatrix}$

$\rightarrow \Delta x = -0.1789\times10^{-3}$ $\Delta y = -0.0167\times10^{-3}$

$\rightarrow \boxed{x_2 = 0.71021 \quad y_2 = 1.51068}$ positive root

first 2020

$\boxed{1}$ a) $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $x_{i+1} = x_i - \frac{x^3-10x^2+5}{3x^2-20x}$

$\rightarrow x_1 = 1.77945$

b) $\% \varepsilon_t = \frac{0.734603 - 1.77945}{0.734603} = 142.2335$

c) $x_2 = 0.973389$ $\% \varepsilon_a = \left| \frac{x^{new} - x^{old}}{x^{new}} \right| \times 100$

$= 82.8098\ \%$

$\boxed{3}$ $f(x) = \ln(x)$ $f'(x) = \frac{1}{x}$

a) $x_0 = 0.7 \rightarrow f'(x_0) = \frac{1}{0.7} = 1.42859$

b) $f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$ $f(x_{i-1}) = f(0.5)$

$= \frac{\ln(0.7) - \ln(0.5)}{0.2} = 1.68236$

~~c) $f'(x_i) \approx \frac{-0.35 - -0.69}{0.2} = \frac{0.30}{0.2} = \boxed{1.7}$~~

b) $\therefore f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{0.2}$

$\rightarrow f(x_i) = 0.2 \cdot f'(x_i) + f(x_{i-1})$

$= -0.40743$

c) $f(x_i) = 0.2 \cdot (1.42) + (-0.69) = 0.28 - 0.69 = \boxed{-0.41}$

d) $f'(x_i) = \frac{1}{0.7}$ True $\rightarrow E_a = -0.25379$

$f'(x_i)\ x = 1.687361183$

$\boxed{4}$ a) -0.0900811

b) $\frac{-20 + \sqrt{400 - 1.44}}{0.4} = \frac{-20 + 19.96}{0.4} = \boxed{-0.1}$

c) $\left| \frac{-0.0900811 - -0.1}{-0.0900811} \right| \times 100 = 11.0111\ \%$

$20 + 30\sqrt{x} - 3x = 0$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{-3x + 30x^{0.5} + 20}{\frac{15}{\sqrt{x}} - 3}$

$x_1 =$ $x_{true} = 112$

$f(x_i) \approx \frac{-0.35 - -0.69}{0.2} = \frac{0.34}{0.2} = \boxed{1.7}$

$f'^{true}(x_i) = \frac{10}{7} \rightarrow E_a = \frac{10}{7} - 1.7 = -0.27143$

Subject Date No.

Q1)

$x \times y = 2.3847 \times 0.1555$

2×0.125

$= \boxed{0.25}$ ✓

$\% RE_t = 32.582\%$

0	1	11	100	
0	1	11	101	
0	1	11	110	
0	1	11	111	
0	1	10	100	0.125
0	1	10	101	0.15625
0	1	10	110	
0	1	10	111	0.21875
0	1	01	100	0.25
0	1	01	101	
0	1	01	110	
0	1	01	111	
0	0	01	100	
0	0	10	100	2
0	0	10	101	2.5
0	0	10	110	3

Q2) $x_0 = 1.01$, $f(x) = \ln(x) - x^2$

a) $f(x_{i+1}) = f(x_i) + f'(x_i)\,h + \frac{f''(x_i)}{2!} h^2$

$f'(x_i) = \frac{1}{x_i} - 2x_i$, $f''(x_i) = \frac{-1}{x^2} - 2$, $f^{(3)}(x_i) = \frac{2}{x^3}$

$\therefore f(x_{i+1}) = -1.01015 - 0.10299 - 0.0149015$

$= -1.1280415$

b) $\therefore \xi = 0.0008593 = R_2 = \frac{f^3(\xi)}{3!} \cdot h^3$

$\rightarrow \frac{2}{\xi^3} \cdot \frac{1}{6} h^3 = \xi$

if $\xi = 2 \rightarrow h = 0.27402$

if $\xi = 1 \rightarrow \boxed{h = 0.13701}$

b)

$f'(1.5) = \frac{f(2) - f(1.5)}{0.5} = \frac{7.915 - 4.347}{0.5} = 7.136$

$\rightarrow \%\varepsilon_t = 18.36$

$f'_{true}(1.5) = 6.029$

7) $\because$ $x_l = 4.5$, $x_u = 6$ $\rightarrow$ $x_r = 5.25$

$f(x_r) \cdot f(x_l) = -1.1211 \cdot -3.6875 > 0$

$\rightarrow$ $x_l = x_r$

$\therefore$ $\boxed{x_r^2 = 5.625}$ $f(x_r) \cdot f(x_l) < 0$ $\rightarrow$ $x_u = x_r$

$\therefore$ $x_r^3 = 5.4375$

8) $\because$ $x_r = x_u - \dfrac{f(x_u)\,(x_l - x_u)}{f(x_l) - f(x_u)}$

$$x_r^1 = 6 - \frac{7\,(4.5-6)}{-3.6875 - 7} = 5.0175$$

$f(x_r) \cdot f(x_l) > 0$ $\rightarrow$ $x_l^{new} = 5.0176$

$$\rightarrow x_r^2 = 6 - \frac{7\,(5.0176-6)}{-1.0014-7} = 6 - \frac{-6.8776}{8.0014}$$

$\rightarrow x_r^2 = 6 - 0.85953$

$= 5.1404$ ✗

9) $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ $f'(x_0) = 3.9104$

$f(x_0) = 0.75141$

$\rightarrow x_{i+1} = 0.3 - 0.19215 = 0.10785$

$f'(x_1) = 6.3672$

$f(x_1) = ~~-22~~ -0.22691$

$$\rightarrow x_2 = 0.10785 + \frac{0.22691}{6.3672}$$

$= 0.10785 + 0.035637$

$= 0.14348$ ✗

10) $x_{i+1} = x_i - \dfrac{\delta x_i\, f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$

$$x_1 = 0.3 - \frac{0.01 \cdot 0.3 \cdot 0.75141}{0.76309 - 0.75141} = 0.3 - \frac{2.2542 \times 10^{-3}}{0.01168}$$

$= 0.3 - 0.19299$

$= 0.10701$

$$x_2 = 0.10701 - \frac{0.01 \cdot 0.10701 \cdot -0.23226}{-0.22544 - -0.23226} = 0.10701 + \frac{2.4854 \times 10^{-4}}{6.82 \times 10^{-3}}$$

$$= 0.10701 + 0.031442 = 0.10345$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{total error} = \left| f'(x_i) - \frac{f(x_{i+1}) - f(x_{i-1})}{h} \right|$$

$$= \left| -0.35669 - \right.$$

Subject numerical methods mid exam Date 17/4/2021 No.

MOHAMMAD SANAD ALTAHER 130806

Q1) $f(x) = 3x^2 + 0.1e^x$

a) $f'(x) = \frac{e^x}{10} + 6x \rightarrow \boxed{f'(0.25) = 1.62840}$

b) $\because f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$, $x_1 - x_0 = 0.2 \rightarrow \boxed{x_1 = 0.45}$

$\rightarrow f(x_0) = f(x_1) - h \cdot f'(x_0)$

$= \boxed{0.43865}$

c) $f(x_0) = (3 \times 0.20) + 0.1 \times (1.56) - 0.2\left[\frac{1.28}{10} + 6 \cdot 0.25\right]$

$= 0.6 + 0.15 - 0.2 \left[1.62\right]$

$= 0.75 - 0.32 = \boxed{0.43}$

D) $\because$ total numerical error $= \left| f'(x_0) - \frac{f(x_1) - f(x_0)}{0.2} \right|$

$= \left| 1.62840 - \frac{0.6 + 0.15 - 0.70}{0.2} \right|$

$= \boxed{0.6216}$

MOHAMMAD SANAD ALTAHER 130806

Q2)

$$\begin{bmatrix} 8 & 1 & -2 \\ 3 & 7 & 1 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$$

Rearranged

a) $x_1^1 = \dfrac{b_1 - a_{12} - a_{13}x_3}{a_{11}} = \dfrac{1 - 1\cdot x_2^0 + 2\cdot x_3^0}{8}$

$\rightarrow \boxed{x_1^1 = 0.25}$ $x_2^1 = \dfrac{-6}{7}$

b) $x_3^1 = \dfrac{b_3 - a_{31}x_1^0 - a_{32}x_2^0}{a_{33}}$

$\rightarrow \boxed{x_3^1 = 0.2222}$

c) $x_1^{(2)} = \dfrac{1 - 1\cdot x_2^1 + 2\cdot x_3^1}{8} = 0.28769$

d) $x_3^{(2)} = \dfrac{8 - 1\cdot x_1^1 - 5\cdot x_2^1}{9} = \boxed{1.3373}$

e) $\left|\dfrac{x_1^2 - x_1^1}{x_1^2}\right| \times 100\% = 13.1\%$

Subject Date No.

MOHAMMAD SANAD ALTAHER 130806

Q3) $f(x) = x^3 - 10x^2 + 5$, $x_1^{exact} = 0.734603$

a) $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $f'(x) = 3x^2 - 20x$

$\rightarrow x_1 = 0.2 - (-1.1876) = \boxed{1.3876}$

b) $\varepsilon_t = \frac{0.734603 - 1.3876}{0.734603} \times 100 = 88.89 \%$

c) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \boxed{0.86053}$

d) $\varepsilon_a = \left| \frac{x_2 - x_1}{x_2} \right| \times 100 = 61.24 \%$

MOHAMMAD SANAD ALTAHER 130806

Q4)

a) $x = \frac{-20 + \sqrt{20^2 - 4(0.2)(1.8)}}{2 \cdot 0.2} = -0.090081$

b) $x = \frac{-20 + \sqrt{400 - 1.44}}{0.4} = \frac{-20 + 19.96}{0.4} = \boxed{-0.1}$

c) $\left|\frac{-0.090081 - -0.1}{-0.090081}\right| \times 100\% = 11.011\%$

Subject Chapter 17: least-squares regression Date 9/5/2021 No.

* $S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$

+ to find a_0 λ a_1 in $y = a_0 + a_1 x + e$:

- derive with respect to a_0 λ a_1

$$\rightarrow \frac{\partial S_r}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i) \quad λ \quad \frac{\partial S_r}{\partial a_1} = -2 \sum (y_i - a_0 - a_1 x_i) x_i$$

- set derivative equal to zero :

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i \quad λ \quad 0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2$$

$$\therefore n a_0 + (\sum x_i) a_1 = (\sum y_i) \quad λ \quad (\sum x_i) a_0 + (\sum x_i^2) a_1 = (\sum x_i y_i)$$

- solve the simultaneous equations :

$$\boxed{a_1 = \frac{n(\sum x_i y_i) - (\sum x_i)(\sum y_i)}{n(\sum x_i^2) - (\sum x_i)^2}}$$

λ

$$\boxed{a_0 = \bar{y} - a_1 \bar{x}}$$, line superscript : mean

example 17.1: ∴ $n = 7$, $\sum x_i \cdot y_i = 119.5$

λ $\sum x_i$ ∴ $\sum y_i = 672$

λ $\sum (x_i)^2 = 140$ λ $(\sum x_i)^2 = 784$

∴ $a_1 = 0.83933$

λ $\bar{y} = \frac{24}{7}$ λ $\bar{x} = \frac{28}{7} \rightarrow a_0 = 0.07143$

* standard error of the estimate: $S_{y/x} = \frac{\sqrt{S_r}}{\sqrt{n-2}} = \sqrt{\frac{S_r}{n-2}}$

* coefficient of determination: define S_t , where t: variable (x or y)

$$\rightarrow S_t = S_y = \sqrt{\frac{\sum (y_i - \bar{y})^2}{n-1}}$$

$$\text{or } S_t = S_x = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

$\therefore$ the coefficient of determination, $r^2 = \frac{S_t - S_r}{S_t}$ $S_t = \sum (t_i - \bar{t})^2$

& r is the correlation coefficient:

or $r = \frac{n \cdot \sum x_i y_i - (\sum x_i)(\sum y_i)}{\sqrt{n \sum x_i^2 - (\sum x_i)^2} \cdot \sqrt{n \sum y_i^2 - (\sum y_i)^2}}$

example 17.2: $\because S_y = \sqrt{\frac{\sum (y_i - \bar{y})^2}{n-1}} = \sqrt{\frac{22.7143}{6}} = 1.94569$

& $S_{y/x} = \sqrt{\frac{S_r}{n-2}}$

$S_r = 0.168679 + 0.562455 + 0.34731 + 0.32646 + 0.5846 + \dots$

$\rightarrow S_r = 2.9911 \therefore S_{y/x} = 0.77345$

$\therefore r^2 = 0.86832 \rightarrow r = 0.931835$

a S_r: sum of squared rests

example from notes: $x_i = \{0, 1, 2.5\}$ & $y_i = \{3, 4.2, 7.1\}$

$\rightarrow \sum x_i = 3.5$, $\sum y_i = 14.3$, $\sum (x_i)^2 = 7.25$

$(\sum x_i)^2 = 12.25$, $\sum x_i y_i = 21.95$

$\therefore \boxed{a_1 = 1.66316}$

$\rightarrow \bar{y} = \frac{14.3}{3}$ & $\bar{x} = \frac{3.5}{3} \rightarrow \boxed{a_0 = 2.826313}$

or by matrices: $\begin{bmatrix} \sum x_i & \sum x_i^2 \\ n & \sum x_i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix}$

- this solution gives the lowest S_r

17.2: polynomial regression

* for $y = a_0 + a_1 x + a_2 x^2 + e$

→ $S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$

- derive with respect to each coefficient then set to zero:

∴ $n a_0 + (\sum x_i) a_1 + (\sum x_i^2) a_2 = \sum y_i$

∧ $a_0 (\sum x_i) + (\sum x_i^2) a_1 + (\sum x_i^3) a_2 = \sum x_i y_i$

∧ $a_0 (\sum x_i^2) + (\sum x_i^3) a_1 + (\sum x_i^4) a_2 = \sum x_i^2 y_i$

$$\therefore \begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix}$$

- this can be extended to m-th order polynomials

→ $s_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$, m: order of polynomial

Example 17.5: $n = 6$, $\sum x_i = 15$, $\sum x_i^2 = 55$

$\sum x_i^3 = 225$, $\sum x_i^4 = 979$

$\sum y_i = 152.6$, $\sum y_i x_i = 585.6$

$\sum y_i x_i^2 = 2488.8$

$\rightarrow a_0 = 2.4786 , a_1 = 2.3593 , a_2 = 1.8607$

$\land S_{y/x} = \sqrt{\frac{3.74657}{6-(2+1)}} = 1.1175$

$\land R^2 = \frac{\sum (y_i - \bar{y})^2 - S_r}{\sum (y_i - \bar{y})^2} = 0.99851$

has been explained

$\therefore$ 99.851% of the original uncertainty is maintained by the model

- the standard error estimate is the standard deviation

- If $S_r = 0$, then $f(x)$ is accurate (perfect fit)
  $\therefore R^2 = 1$
- if $S_r = S_t$, then no benefit is gained by doing regression.
  $\rightarrow R^2 = 0$
- $0 \leq R^2 \leq 1$ $\land$ a larger R^2 is desired

example from notes : $S_r = 0.1274 \rightarrow S_{y/x} = \sqrt{\frac{0.1274}{n-2}} = \boxed{0.3569}$

$\land R^2 = \frac{\sum (y - \bar{y})^2 - S_r}{\sum (y - \bar{y})^2} = 0.9857$

* linearization of nonlinear relationships :

(1) exponential : $y = a_1 e^{B_1 x} \rightarrow \ln(y) = \ln(a_1) + B_1 x$

$\therefore \ln(y) = Z \land \ln(a_1) = a_0 \land B_1 = a_1$

$\rightarrow \ln(y) \ln(a_1) + B_1 x \rightarrow Z = a_0 + a_1 x$

$\rightarrow$ straight line with intercept

$$\rightarrow \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum Z_i \\ \sum Z_i x_i \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} \overset{\text{from initial}}{\ln(a_1)} \\ B_1 \end{bmatrix} = \begin{bmatrix} \sum \ln(y_i) \\ \sum \ln(y_i) \cdot x_i \end{bmatrix}$$

example on exponential linearization: $n=3$

$$\begin{bmatrix} 3 & 2.5 \\ 2.5 & 3.25 \end{bmatrix}\begin{bmatrix} \ln(\alpha_1) \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 2.9321 \\ 4.5526 \end{bmatrix}$$

$\therefore \ln(\alpha_1) = -0.52919$, $\beta_1 = 1.8079$

$\rightarrow Z = 1.8079 - 0.52919 \cdot x$ or $y = 0.5891\, e^{1.8079 \cdot x}$

② power model: $y = \alpha x^{\beta}$ $\rightarrow \log y = \beta_2 \log(x) + \log(\alpha)$

$$Z = a_1 h + a_0$$

$$\therefore \begin{bmatrix} n & \sum h_i \\ \sum h_i & \sum h_i^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum Z_i \\ \sum Z_i h_i \end{bmatrix}$$

then take 10^{h_i} to find x_i and 10^{a_0} to find α, etc...

③ growth-saturation model: $\boxed{y = \alpha \dfrac{x}{\beta + x}}$

- invert to linearize : $\dfrac{1}{y} = \dfrac{\beta}{\alpha x} + \dfrac{1}{\alpha}$

$a_1 = \dfrac{\beta}{\alpha}$, $a_0 = \dfrac{1}{\alpha}$

$$Z = a_1 h + a_0$$

$$\rightarrow \begin{bmatrix} n & \sum h_i \\ \sum h_i & \sum h_i^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum Z_i \\ \sum Z_i x_i \end{bmatrix}$$

$\rightarrow \alpha = \dfrac{1}{a_0}$, $\beta = \dfrac{a_1}{a_0}$, $\dfrac{1}{y}$, $\dfrac{1}{x}$ are linearly related

general linear least squares:

for a function: $y = a_0 z_0 + a_1 z_1 + a_2 z_2 + \cdots + a_m z_m + e$

where z includes: $z_0 = 1, z_1 = x_1, z_2 = x_2, \ldots, z_m = x_m$

or $z_0 = x^0, z_1 = x^1, z_2 = x^2, \ldots, z_m = x^m$

or sinusoidal: $z_0 = 1$, $z_1 = \cos(wt)$, $z_2 = \sin(wt)$

- however, $f(x) = a_0(1 - e^{-a_1 x})$ can't be put in the above format hence it is nonlinear.

$$\vec{y} = Z\vec{a} + \vec{e}$$

- in general: $\{Y\} = [Z]\{A\} + \{E\}$

& $$[Z] = \begin{bmatrix} z_{01} & z_{11} & \cdots & z_{m1} \\ z_{02} & z_{12} & \cdots & z_{m2} \\ \vdots & \vdots & & \vdots \\ z_{0n} & z_{1n} & \cdots & z_{mn} \end{bmatrix}$$

- m: number of variables or degree and n: number of data points

example:

① linear/straight line: $y = a_0 + a_1 x \Rightarrow z_1 = x$ & $z_0 = 1$

② quadratic polynomial: $y = a_0 + a_1 x + a_2 x^2 \rightarrow z_0 = 1, z_1 = x, z_2 = x^2$

∵ $S_r = \sum_i e_i^2 = (\vec{e}^T \vec{e}) = \|\vec{e}\|^2$ (the norm squared)

∴ $S_r = \vec{a}^T Z^T Z \vec{a} - 2\vec{y}^T Z \vec{a} + y^T y$

→ $(Z^T Z)\vec{a} = y^T Z$ or $\boxed{\vec{a} = (Z^T Z)^{-1} y^T Z}$

example from notes: $z_0 = x$, $z_1 = e^{-x}$ → $y = a_0 x + a_1 e^{-x}$

x	0	0.5	1.2
y	0.25	0.7	2

$$\rightarrow [Z] = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.606531 \\ 1.2 & 0.301194 \end{bmatrix}$$

$$Z^T Z = \begin{bmatrix} 1.69 & 0.6646983 \\ 0.6646983 & 1.4584968 \end{bmatrix}$$

$$\rightarrow (Z^T Z)^{-1} = \begin{bmatrix} 0.7209337559 & -0.3285370933 \\ -0.3285370933 & 0.8353098196 \end{bmatrix}$$

$$y^T Z = \begin{bmatrix} 2.75 \\ 1.27696 \end{bmatrix} \rightarrow \{a\} = \begin{array}{c} a_0 \\ a_1 \end{array} \begin{bmatrix} 1.56304 \\ 0.16318 \end{bmatrix}$$

→ best fit: $y = 1.563 \cdot x + 0.1632 \cdot e^{-x}$

* nonlinear regression:

given $y_i = f(x_i) + e_i$

using taylor series: $f(x_i)_{j+1} = f(x_i)_j + \frac{\partial f(x_i)_j}{\partial a_0} \Delta a_0 + \frac{\partial f(x_i)_j}{\partial a_1} \Delta a_1 - -$

$$\therefore \boxed{y_i - f(x_i)_j = \frac{\partial f(x_i)_j}{\partial a_0} \Delta a_0 + \frac{\partial f(x_i)_j}{\partial a_1} \Delta a_1 + \text{error}}$$

$$\therefore \{D\} = [Z_j]\{\Delta A\} + \{E\}$$

$$\{D\} = \begin{Bmatrix} y_1 - f(x_1) \\ y_2 - f(x_2) \\ \vdots \\ y_n - f(x_n) \end{Bmatrix} \quad \{\Delta A\} = \begin{Bmatrix} \Delta a_0 \\ \Delta a_1 \\ \vdots \\ \Delta a_n \end{Bmatrix} \quad [Z_j] = \begin{bmatrix} \frac{\partial f_1}{\partial a_0} & \frac{\partial f_1}{\partial a_1} \\ \frac{\partial f_2}{\partial a_0} & \frac{\partial f_2}{\partial a_1} \\ \vdots & \vdots \\ \frac{\partial f_n}{\partial a_0} & \frac{\partial f_n}{\partial a_1} \end{bmatrix}$$

$$\therefore \ \{\Delta A\} = \left[[Z_j]^T[Z_j]\right]^{-1}\left\{[Z_j]^T\{D\}\right\}$$

$\wedge \quad a_{0,j+1} = a_{0,j} + \Delta a_0 \ , \ a_{1,j+1} = a_{1,j} + \Delta a_1$

$$\wedge \ |\varepsilon_a|_k = \left|\frac{a_{k,j+1} - a_{k,j}}{a_k}\right| \times 100\%$$

practice problems:

17.3: slope = 0.352469 $\wedge$ intercept = 4.851435

$\wedge$ $R = 0.91477 \rightarrow R^2 = 0.8368$

$\wedge$ $S_r = (1 - R^2)\, S_t \quad \wedge \ S_t =$ from calc.

$\bar{y} = 8.2 \rightarrow S_t = S_y = 55.56 \rightarrow S_r = 9.0674$

or $\sum(x_i \cdot y_i) = \cancel{84}911 \ \wedge \ \sum x_i \sum y_i = 82 \cdot 95 = 7790$

$\sum x_i = 95 \rightarrow \bar{x} = 9.5 \ \wedge \ (\sum x_i)^2 = 9025$

$\sum x_i^2 = 1277 \quad \therefore \ a_1 = 0.08945 \times \quad 0.3525$

$\wedge \ a_0 = 7.3882 \times \quad 4.85143$

$\rightarrow S_r =$

17.8: $\log(y) = \underset{a_1}{b_2} \log x + \underset{a_0}{\log a_2}$

$$\rightarrow \begin{bmatrix} n & \sum h_i \\ \sum h_i & \sum h_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum z_i \\ \sum z_i h_i \end{bmatrix}$$

$\sum h_i = \sum \log x_i = 7.11126 \qquad \sum h_i^2 = 7.138402$

$\sum z_i = 8.32993 \ \wedge \ \sum z_i h_i = 7.1370845$

$\therefore \ a_0 = 1.325224 \ \wedge \ a_1 = -0.5402903$

$\rightarrow \ y = 21.1458\, x^{-0.54029}$

$x = 9 \rightarrow \boxed{y = 6.4514}$

17.9 : $$\begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_i Z_i \\ \sum_i Z_i x_i \end{bmatrix}$$

$\sum_i Z_i = \sum_i \ln(y_i) = 44.617$, $\sum_i x_i = 8.3$, $\sum_i x_i^2 = 14.09$

$\sum_i Z_i \cdot x_i = 63.8555 \wedge n = 6$

$\rightarrow a_0 = 6.3039 \wedge a_1 = 0.818652$

$y = a e^{\beta x}$, $\beta = a_1 = 0.818652$

$\wedge\ a = e^{6.3039} = 546.6$

17.15 : $y = a + bx + \frac{c}{x} = yx = ax + bx^2 + c$

$\rightarrow Z = a_0 + a_1 x + a_2 x^2$

$\sum_i x_i = 15$, $\sum_i x_i^2 = 55$ $\sum_i x_i^3 = 225$

$\wedge \sum_i x_i^4 = 979$, $\sum Z_i = 18.6$, $\sum y_i x_i = 64.1$

$\sum y_i \cdot x_i^2 = 255.3$

$\sum y_i \cdot x_i^3 = 1097.3$

$$\begin{bmatrix} 5 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 18.6 \\ 64.1 \\ 255.3 \end{bmatrix}$$

$\rightarrow a_0 = 1.02$, $a_1 = 0.2142867$, $a_2 = 1.014283$

$\therefore y = 0.214286 + 1.014285 x + \frac{1.02}{x}$

Subject quiz 1 practice Date 16/5/2021 No.

Q₁) $f(x) = 0.5x + 3\sqrt{x}$

a) 2nd order taylor : $f(1.58) = 0.5(1.08) + 3\sqrt{1.08} + \left(0.5 + \frac{3}{2\sqrt{x}}\right)0.5$

$+ 0.5 \cdot (0.5)^2 \cdot \left(\frac{-3}{4x^{3/2}}\right.$

$\rightarrow f(1.58) \approx 4.62938 - \frac{1}{8} \cdot \frac{3}{4x^{3/2}} = 4.54585$

b) $f_{exact} = 4.5609415 \rightarrow$ Remainder $= 0.01509$

$$R_2 = \frac{f^{(3)}(\xi)}{(n+1)!}(0.5)^3 = \frac{1}{6} \cdot \frac{1}{8} \cdot \frac{9}{8\xi^{5/2}} = \frac{3}{128} \cdot \xi^{-5/2}$$

$\because \xi \in [1.08, 1.58]$, assume $\xi = 1.35$

$\rightarrow R_2 = 0.01107 \times \xi \neq 1.35$

Q₂) $x_0 = 4.25$, $y_0 = 1$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 0.5$

$\frac{\partial v}{\partial x} = -3e^{-x}$, $\frac{\partial v}{\partial y} = 1$

$$\rightarrow \begin{bmatrix} 2x & 0.5 \\ -3e^{-x} & 1 \end{bmatrix} \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix} = \begin{bmatrix} -u_i \\ -v_i \end{bmatrix}$$

at $i = 0 \rightarrow \begin{bmatrix} 8.5 & 0.5 \\ -0.042793 & 1 \end{bmatrix} \begin{bmatrix} x_{i+1} - 4.25 \\ y_{i+1} - 1 \end{bmatrix} = \begin{bmatrix} -13.5625 \\ +0.957209 \end{bmatrix}$

$\rightarrow x_1 = 2.602242 \wedge y_1 = 1.886694$

at $i = 1 \rightarrow \begin{bmatrix} 5.204484 & 0.5 \\ -0.222322 & 1 \end{bmatrix} \begin{bmatrix} x_{i+1} - 2.602242 \\ y_{i+1} - 1.886694 \end{bmatrix} = \begin{bmatrix} -2.91501 \\ -0.109016 \end{bmatrix}$

$\rightarrow x_2 = 2.101938 \wedge y_2 = 1.6664$

$h = 2.0799$ $y = 0.1891$

Q1) 0 0 0 0 1.0 0

0 1 1 0 1 0 0 0.125

$\boxed{1\ 0\ 1}$ $0.15625 = \cancel{y}$

$\boxed{1\ 1\ 0}$ $0.1875 = y$

0 0 1 0 1 0 0 $2 = h$

1 0 1 2.5

→ $h \times y = 0.375$

→ $\boxed{h \times y = 0101\ 110}$

$h \times y$ True = 4.666 %

Q2) $f(x) = \ln(x) - x^2$, $x_0 = 3.89$

$$\to f(x_1) = \ln(3.89) - (3.89)^2 + \left(\frac{1}{3.89} - 2\cdot 3.89\right)0.1 + \frac{(0.1)^2}{2}\cdot\left(-\frac{1}{(3.89)^2} - 2\right)$$

$$= -14.53631432$$

$$\therefore e_{abs} = R_2 = \frac{f^{(3)}(\xi)}{3!}(0.1)^3 \quad , \quad f^{(3)}(\xi) = \left|\frac{2}{x^3}\right|$$

$$\to \frac{h^3}{3x^3} \quad , \quad x^3 = \xi^3$$

$$\therefore \xi \in [1, 2] , \quad h^3 \le 3\xi^3 \cdot \varepsilon$$

$$\to h \le \xi \sqrt[3]{3\varepsilon} \quad , \quad \varepsilon = 0.0005026$$

$$\to h \le \xi \cdot 0.1146694985$$

$$\to h \le 0.1146694985$$

Q3) $E_a^{(n)} = \frac{\Delta x^0}{2^n}$ $n = \log_2\left(\frac{\Delta x^0}{E_a^n}\right)$

$\Delta x^0 = 6.1129$

Q1) $f(x) = x^3 - 10x^2 + 5$

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $f'(x_i) = 3x^2 - 20x$

a) $x_1 = 0.15 - -1.62945 = 1.77945$

b) $\frac{0.734603 - 1.77945}{0.734603} = \%142.23\%$

c) $x_2 = 0.773389$

d) 82.81

Q2) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$

$f(1.6) = \frac{10}{3} + \cancel{\frac{-1}{0.3}} + \cancel{\frac{1}{0.3}} = 3.33333$

a) 3.3333 b) -1.6669 ✓ c) d) 1.35672

$\frac{1}{0.3} - \cancel{\frac{1}{0.3}} + \cancel{\frac{1}{0.3}}$

c) $\therefore R_2 = -1.666669 = \frac{f^{(3)}(\xi)}{3!}h^3$

$\Rightarrow$ ~~$\xi = 1.55836$~~ $\xi = 1.356962$

Q3) b) $f'(x_0) = \frac{f(x_0) - f(0.5)}{0.2} = 1.68236$

$\frac{-0.35 + 0.69}{0.2} = \boxed{1.7}$

$f'(x_i) = \frac{-0.10 + 0.69}{0.4} = 0.225$

Subject mid exam Redone Date 17/5/2021 No.

Q1) $\frac{-30 + \sqrt{30^2 - 4(0.3)(0.8)}}{2 \cdot 0.3}$ $\frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$

a) -0.026674 or

b) $\frac{-30 + \sqrt{900 - 0.96}}{0.6} = \frac{-30 + 29.98}{0.6} = -0.03$

or subtractive cancellation: $\frac{-1.6}{30 + 29.98} = -0.02$

c) 12.49 % or 25.02%

Q2) a) $f(x) = x^3 - 10x^2 + 5$, $f(x_0) = 4.901$

$x_1 = 0.1 - \frac{f(x_0)}{3x^2 - 20x} = 2.58982$

b) 252.27 %

c) $x_2 = 2.58982 - 1.409652 = 1.178168$

d) $\%R_a = 119.64\%$

Q3) Rearrange:

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 7 & 1 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$\rightarrow x_1^1 = \frac{2 - 1 + 1}{4} = \frac{1}{2}$

$x_2^1 = \frac{3 - 3 - 1}{7} = \frac{-1}{7}$

$x_3^1 = \frac{5 - 1 - 5}{9} = \frac{-1}{9}$

$\rightarrow x_1^2 = \frac{2 + \frac{1}{7} - \frac{1}{9}}{4} = 0.50794$

$\rightarrow x_3^2 = \frac{5 - \frac{1}{2} + \frac{5}{7}}{9} = 0.57936$

$\%RE_{approx} = \frac{0.50794 - 0.5}{0.50794} = 1.563\%$

Q4 $f'(x) = 6x + 0.1e^x$

a) $f'(0.25) = 1.6284$

b) $f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{0.2} = \frac{0.448428}{0.2} = 2.24214$

c) $\frac{0.6 + 0.15 - 0.18 - 0.12}{0.2} = \frac{0.45}{0.2} = 2.25$

d) total error $= \left| f'(x_i) - \frac{f(x_{i+1}) - f(x_i)}{0.2} \right|$

$\rightarrow |1.6284 - 2.25| = 0.6216$

Subject quiz Date 17/5/2021 No.

MOHAMMAD SANAD ALTAHER 130806

Q1) $f(x) = 3x^2 + 0.35 e^x$

a) $f'(x) = 6x + 0.35 e^x$

$f'(0) = 0.35$

b) $f'_{approx}(x_0) = \frac{f(x_0) - f(x_0 - 0.2)}{0.2}$

$= \frac{0.35 - 0.40666}{0.2}$

$= -0.28278$

c) $f'_{approx}(x_0) = \frac{0.4 - 0.4}{0.2} = 0$

d) total numerical error $= |f'_{exact}(x_0) - f'_{approx}(x_0)|$

$\rightarrow 0.35 - 0 = 0.35$

MOHAMMAD SANAD ALTAHER 130806

rearrange

$$\begin{bmatrix} 8 & 1 & -5 \\ 3 & 6 & 1 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$$

$$x_1^1 = \frac{1 - 1 + 5}{8} = 0.625$$

$$x_2^1 = \frac{3 - 3 \cdot 0.625 - 1}{6} = 0.0208333$$

$$x_3^1 = \frac{10 - 0.625 - 5 \cdot 0.020833}{9} = 1.03009$$

$$x_1^2 = \frac{1 - 0.020833 + 5 \cdot 1.03009}{8} = \cancel{0.7}$$

$$= \boxed{0.76620}$$

Subject numerical methods Date 2/6/2024 No.

* Interpolation:

* Quadrati interpolation:

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

$$\rightarrow b_0 = f(x_0) \quad , \quad b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\rightarrow b_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}$$

example 18.2:

$x_0=1$, $x_1=4$, $x_2=6$ & $f(x_0)=0$, $f(x_1)=1.386294$

$f(x_2) = 1.791759$

$\rightarrow b_0 = 0$, $b_1 = \frac{1.386294}{3} = 0.462098$

& $b_2 = -0.0518731$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$a_0 = b_0 - b_1 x_0 + b_2 x_0 x_1$$

$$a_1 = b_1 - b_2 x_0 - b_2 x_1 \qquad a_2 = b_2$$

* general form of newton's interpolating polynomials:

$$f_n(x) = b_0 + b_1(x-x_0) + \cdots + b_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

s.t. $b_0 = f(x_0)$, $b_1 = f[x_1, x_0]$ (same as above)

$b_2 = f[x_2, x_1, x_0]$, $b_n = f[x_n, x_{n-1}, \cdots, x_1, x_0]$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

example 18.3:

$$f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$$

$\rightarrow b_0 = 0$, $b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{} = 0.462098$

$\rightarrow b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$

$f[x_2, x_1] = \frac{1.791759 - 1.386294}{6-4} = 0.2027325$

$\rightarrow f[x_2, x_1, x_0] = -0.0518731$

$\rightarrow b_3 = f[x_3, x_2, x_1, x_0]$

$$f[x_n, x_{n-1}, \dots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \dots x_1] - f[x_{n-1}, x_{n-2}, \dots x_0]}{x_n - x_0}$$

$\rightarrow b_3 = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$

$\rightarrow f[x_3, x_2, x_1] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1}$

$\rightarrow f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = 0.182321$

$\rightarrow f[x_3, x_2, x_1] = -0.0204115$

$\rightarrow f[x_2, x_1, x_0] = -0.0518731$

$\rightarrow b_3 = 7.8654 \times 10^{-3}$

* errors of newton's interpolating polynomials :

$$R_n = f[x, x_n, x_{n-1}, \dots, x_0] \cdot (x-x_0) \cdots (x-x_n)$$

$\rightarrow$

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \dots, x_0] \cdot (x-x_0) \cdots (x-x_n)$$

as long as an extra point $f(x_{n+1})$ is available.

$$R_n = f_{n+1}(x) - f_n(x)$$

& lagrange interpolating polynomials:

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

$$\text{s.t. } L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad (\Pi : \text{product of})$$

$\therefore$ for $n=2$:

$$f_2(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\cdot\left(\frac{x-x_2}{x_0-x_2}\right)\cdot f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)\cdot f(x_1)$$

$$+ \left(\frac{x-x_0}{x_2-x_0}\right)\cdot\left(\frac{x-x_1}{x_2-x_1}\right)\cdot f(x_2)$$

example 18.6:

first order: $\left(\frac{2-4}{1-4}\right)\cdot 0 + \left(\frac{2-1}{4-1}\right)\cdot 1.386294 = 0.462098$

second order: $\frac{(2-4)\cdot(2-6)}{(1-4)(1-6)}\cdot 0 + \left(\frac{2-1}{4-1}\right)\left(\frac{2-6}{4-6}\right)\cdot 1.386294$

$+ \left(\frac{2-1}{6-1}\right)\left(\frac{2-4}{6-4}\right)\cdot 1.791760 = 0.56584$

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0]\cdot\prod_{i=0}^{n}(x - x_i)$$

- the lagrange method is just a reformulation of newton's method therefore it gives the same results. however, it can be used to calculate the coefficients simultaneously, whereas the newton method can only calculate them in series (i.e., b_0 first, then b_1 $\rightarrow$)

example from notes: $x_0 = 0$, $f(x_0) = 1$, $x_1 = 1$, $f(x_1) = 0.368$

$x_2 = 1.5$, $f(x_2) = 0.223$

(1) newton's method:

$b_0 = 1$, $b_1 = -0.632$, $b_2 = 0.24133$

② lagrange method:

$$f_2(x) = L_0(x) f_0(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$L_0(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\cdot\left(\frac{x-x_2}{x_0-x_2}\right), \quad L_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2}$$

& $L_2 = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1}$

$$\rightarrow f_2(x) = \frac{1}{1.5}(x-1)(x-1.5)\cdot 1 - 2\cdot(x-1.5)\cdot x\cdot 0.368$$
$$+ \frac{1}{0.75}\cdot(x)\cdot(x-1)\cdot 0.223$$

* Spline interpolation:

- linear splines: $f(x) = f(x_0) + m_0(x-x_0)$, $x_0 \leq x \leq x_1$

$f(x) = f(x_1) + m_1(x-x_1)$, $x_1 \leq x \leq x_2$

$\downarrow$ $\downarrow$

$f(x) = f(x_{n-1}) + m_{n-1}(x-x_{n-1})$, $x_{n-1} \leq x \leq x_n$

$$\therefore m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

\+ conditions for quadratic splines:

1- the function values of adjacent polynomials must be equal at the interior knots:

$$a_{i-1} x_{i-1}^2 + b_{i-1} x_{i-1} + c_{i-1} = f(x_{i-1})$$

& $a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$

2- the first and last functions must pass through the end points:

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$

3- the first derivatives at interior knots must be equal:

$$2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i$$

4- The second derivative at the first point should be zero: $a_1 = 0$

example 18.9:

$a_0(3)^2 + b_0(3) + c_0 = 2.5$, $a_1(4.5)^2 + b_1(4.5) + c_1 = 1$

$a_2(7)^2 + b_2(7) + c_2 = 2.5$, $a_3(9)^2 + b_3(9) + c_3 = 0.5$

condition 1: $9a_0 + 3b_0 + c_0 = \cancel{9a_1 + 3b_1 + c_1}$ cancelled by ②

$20.25a_1 + 4.5b_1 + c_1 = 20.25a_2 + 4.5b_2 + c_2$ ②

$49a_2 + 7b_2 + c_2 = 49a_3 + 7b_3 + c_3$ ③

condition 2: $9a_1 + 3b_1 + c_1 = 2.5$

& $81a_3 + 9b_3 + c_3 = 0.5$

condition 3: $2(4.5)a_1 + b_1 = 2(4.5)a_2 + b_2$

& $14a_2 + b_2 = 14a_3 + b_3$

condition 4: $a_1 = 0$

$\rightarrow 3b_1 + c_1 = 2.5$ & $4.5b_1 + c_1 = 1$

$\therefore b_1 = -1$ & $c_1 = 5.5$

& $9a_2 + b_2 = -1$, $20.25a_2 + 4.5b_2 + c_2 = 1$

& $49a_2 + 7b_2 + c_2 = 2.5$

$\rightarrow a_2 = 0.64$, $b_2 = -6.76$, $c_2 = 18.46$

$$\left.\begin{aligned} &49a_3 + 7b_3 + c_3 = 2.5 \\ &14a_3 + b_3 = 2.2 \\ &81a_3 + 9b_3 + c_3 = 0.5 \end{aligned}\right\} \rightarrow \begin{aligned} a_3 &= -1.6 \\ b_3 &= 24.6 \\ c_3 &= -91.3 \end{aligned}$$

$\therefore f_1(x) = -x + 5.5$, $3 \leq x \leq 4.5$

$f_2(x) = 0.64x^2 - 6.76x + 18.46$, $4.5 < x \leq 7$

$f_3(x) = -1.6x^2 + 24.6x - 91.3$, $7 \leq x \leq 9$

example from notes:

cond 1 $a_1 + b_1 + c_1 = a_2 + b_2 + c_2$

$a_2 \cdot 2.25 + b_2 \cdot 1.5 + c_2 = a_3 \cdot 2.25 + b_3 \cdot 1.5 + c_3$

condition 2: $a_1 \cdot 0 + 0 b_1 + c_1 = 3.5 \rightarrow \boxed{c_1 = 3.5}$

$\wedge\ a_3 \cdot 4 + 2b_3 + c_3 = 5$

condition 3: $2a_1 + b_1 = 2a_2 + b_2$

$3a_2 + b_2 = 3a_3 + b_3$

condition 4: $\boxed{a_1 = 0} \rightarrow b_1 + c_1 = 4.5 \rightarrow \boxed{b_1 = 1}$

$\rightarrow 2a_2 + b_2 = 1 \ \wedge\ a_2 + b_2 + c_2 = 4.5$

$\wedge\ 2.25 a_2 + 1.5 b_2 + c_2 = 6$

$\rightarrow \boxed{a_2 = 4}, \boxed{b_2 = -7, c_2 = 7.5}$

$\rightarrow 2.25 a_3 + 1.5 b_3 + c_3 = 6 \ \wedge\ 4a_3 + 2b_3 + c_3 = 5$

$\wedge\ 3a_3 + b_3 = 5 \rightarrow a_3 = -14, b_3 = 47, c = -33$

$\therefore f_1(x) = x + 3.5, \quad 0 \leq x \leq 1$

$f_2(x) = 4x^2 - 7x + 7.5, \quad 1 \leq x \leq 1.5$

$f_3(x) = -14x^2 + 47x - 33, \quad 1.5 \leq x \leq 2$

* newton Cotes integration:

- replace complicated function with approximating function that is easy to integrate:

$$\int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

where $f_n(x) = a_0 + a_1 x + \dots + a_n x^n$

* The Trapezoidal rule: polynomial is first-order (straight line)

$$\rightarrow f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

→ integration: $(b-a)\,\frac{f(a)+f(b)}{2}$

- estimate of local truncation error after single application of trapezoidal rule:

$$E_t = \frac{-1}{12} f''(\xi)(b-a)^3$$

s.t. $\xi \in [a,b]$ (interval)

hence, if the function is linear, f'' will be zero and the integration will be exact.

example 21.1: $a=0$, $b=0.8$, $f(a)=0.2$, $f(b)=0.232$

$$\rightarrow I = \frac{0.8}{2} \cdot 0.432 = 0.1728$$

$$E_t = \text{exact} - I = 1.467733$$

$$\therefore E_t = \frac{-1}{12} f'' \cdot 0.512 \quad \wedge \; f'' = 8000x^3 - 10800x^2 + 4050x - 400$$

average f'' : $\frac{1}{b-a}\int_a^b f''\,dx = -60$

$$\rightarrow E_a = \frac{-1}{12}\cdot(-60)\cdot(0.8)^3 = 2.56$$

- multiple application of trapezoidal rule:
  - split into multiple segments each with the same width.

$$h = \frac{b-a}{n}$$

$$\therefore \quad I = \underbrace{(b-a)}_{\text{width}} \cdot \underbrace{\frac{f(x_0) + \left[2\sum_{i=1}^{n-1} f(x_i)\right] + f(x_n)}{2n}}_{\text{average height}}$$

where n is the number of segments

$$\rightarrow E_a = -\frac{(b-a)^3}{12n^3}\cdot\sum_{i=1}^{n} f''(\xi_i)$$

$$\therefore \underbrace{\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \cong \bar{f}''}_{\therefore \text{ average}} \rightarrow E_a = -\frac{(b-a)^3}{12n^2}\,\bar{f}''$$

if $n=4 \rightarrow x_1 = 0.2, x_2 = 0.4 \ldots$

if $n=3 \rightarrow x_1 = x_0 + 0.2$

$x_1 = x_0 + 0.4, \quad x_0 = 0$

example 21.2:

$$I = 0.8 \cdot \frac{0.2 + 2(2.456) + 0.232}{4}$$

$$\rightarrow I = 1.0688$$

example from notes: $f(x) = e^x - 2x, \quad a=1, b=2$

$$\int_1^2 e^x - 2x \, dx = 1.6907743$$

$$I \simeq (b-a) \frac{f(a)+f(b)}{2} = \frac{e+e^2-6}{2} = 2.053664$$

error $= 0.382894$

example from notes: same as previous but two segments:

$$I = \frac{e - 2 + 2 \cdot e^{1.5} - 4 \cdot 1.5 + e^2 - 4}{4} = 1.767679$$

$$\rightarrow \text{error} = 0.096905$$

& Simpson's $\frac{1}{3}$ Rule:

- use a second order polynomial to approximate the variable:

$$\int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

$$I \cong \underbrace{(b-a)}_{\text{width}} \cdot \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{average height}}$$

$$\& \quad E_t = -\frac{(b-a)^5}{2880} f^4(\xi)$$

or in terms of step size, assuming $h = \frac{(b-a)}{2}$

$$\rightarrow I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\& \quad E_t = -\frac{1}{90} h^5 f^4(\xi)$$

example 21.4:

$$I = 0.8 \cdot \frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

$$\wedge\ E_a = \frac{-(0.8)^5}{2880}\underbrace{\left[\frac{1}{0.8}\int_0^{0.8} -21600+48000x\,dx\right]}_{\text{average fourth derivative}} = \frac{-(0.8)^5}{2880}\cdot(-2400)$$

$$= 0.2730667$$

* The multiple-application simpson's 1/3 rule:

$h = \frac{b-a}{n}$, n = number of segments, all with same width

$$\rightarrow I \cong \underbrace{(b-a)}_{\text{width}} \cdot \underbrace{\frac{f(x_0) + \left[4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j)\right] + f(x_n)}{3n}}_{\text{average height}}$$

$$\rightarrow E_a = -\frac{(b-a)^5}{180\,n^4}\,\bar{f}^{(4)}$$

example 21.5: $n = 4 \rightarrow h = 0.2$

$f(0) = 0.2$, $f(0.2) = 1.288$, $f(0.4) = 2.456$

$f(0.6) = 3.464$, $f(0.8) = 0.232$

$$\rightarrow I = 0.8 \cdot \frac{1}{12}\left[0.2 + 4(1.288 + 3.464) + 2 \cdot 2.456 + 0.232\right]$$

$$\rightarrow I = 1.623467$$

$$\rightarrow E_t = \frac{-(0.8)^5}{180 \cdot 4^4} \cdot -2400 = 0.01709$$

* simpson's 3/8 rule:

- Use a cubic function to approximate:

$$\int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

$$\rightarrow \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right],\quad h = \frac{b-a}{3}$$

$$\therefore \; I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{average height}}$$

↳ $E_t = -\frac{(b-a)^5}{6480} f^4(\xi)$

example from notes: $f(x) = e^x - 2x$, $\frac{1}{3}$ rule, $b=2$, $a=1$

↳ $f(x_0) = 0.7182818$, $f(x_1) = 1.48168907$

↳ $f(x_2) = 3.3890561$

$$I = \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} = 1.672349$$

example from notes: $h = \frac{2-1}{3}$ ∴ $\frac{3}{8}$ rule

$f(x_0) = 0.7182818$, $f(x_1) = 1.12700123$, $f(x_2) = 1.961157$, $f(x_3) = 3.389056$

↳ $I = \frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8} = 1.67147657$

Chapter 21 practice problems:

21.1: $\int_0^{\pi/2} (8 + 4\cos x)\,dx$

a) $4\pi + 4$

b) $I \cong (b-a)\cdot\frac{f(a)+f(b)}{2} = \frac{\pi}{2}\cdot\frac{12+8}{2} = 5\pi$

c) $n=2$ $\frac{\pi}{2}\cdot\left[\frac{12+2(8+4\frac{\sqrt{2}}{2})+8}{4}\right] = \frac{36+4\sqrt{2}}{8}\cdot\pi = 16.3596$

$n=4$

$$\frac{\pi}{2}\left[\frac{12+2\sum_{i=1}^{3} f(x_i) + 8}{8}\right]$$

$$2\sum_{i=1}^{3} f(x_i) = 2\left[8+4\cos\left(\frac{\pi}{8}\right)+8+4\cos\left(\frac{\pi}{4}\right)+8+4\cos\left(\frac{3\pi}{8}\right)\right.$$

$= 64.10935$

$\rightarrow I \cong 16.51483$

d) $\frac{\pi}{2}\cdot\frac{12+8+4(8+4\frac{\sqrt{2}}{2})}{6} = \frac{\pi}{12}\cdot[52+8\sqrt{2}] = 16.5755$

e) $n=4$:

$$\frac{\pi}{2}\cdot\frac{12+8+\left[4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4}^{n-2} f(x_j)\right]}{12}$$

$\rightarrow \sum_{i=1,3,5\ldots}^{3} f(x_i) = 21.22625$

$\sum_{2}^{2} f(x_i) = 8+2\sqrt{2}$

$\rightarrow I \cong 16.56690798$

f) $\frac{\pi}{2}\cdot\frac{12+8+3(8+2\sqrt{3})+(8+2)\cdot 3}{8} = 16.5703918$

21.3: a) $\left[x - \frac{x^2}{2} - x^4 + \frac{1}{3}x^6\right]_{-2}^{4} = \frac{3316}{3} - \frac{4}{3} = \boxed{1104}$

b) $6\cdot\frac{1789+(-29)}{2} = 5280$

c) $n=2$: $\frac{6}{4}\cdot[1760+(-2\cdot 2)] = 2634$

$n=4$: $\frac{6}{8}\cdot[1760+2(1.9375+-2+131.3125)]$

$= 1516.875$

d) $6 \cdot \frac{1760 + 4 \cdot (-2)}{6} = 1752$

e) $6 \cdot \frac{1760 + 3(1) + 31}{8} = 1353.25$

21.13:

a) $\int_0^{0.6} 2e^{-1.5x}\, dx = 0.791240$

b) $0.6 \cdot \frac{2 + 2e^{-0.9}}{2} = 0.843942$

$$\left[0.05 \cdot \frac{1.8555 + 2}{2}\right] + \left[0.1 \cdot \frac{1.597 + 1.8555}{2}\right] + \left[0.1 \cdot \frac{1.597 + 1.3946}{2}\right] + \left[0.1 \cdot \frac{1.3946 + 1.1831}{2}\right] + \left[0.125 \cdot \frac{1.1831 + 0.9808}{2}\right] + \left[0.125 \cdot \frac{0.8131 + 0.9808}{2}\right]$$

$$= \boxed{0.79284}$$

c) use simpson's rule for equally spaced points

$$\rightarrow I \cong \left[0.05 \cdot \frac{1.8555 + 2}{2}\right] + \left[0.3 \cdot \frac{[1.8555 + 3 \cdot 1.597 + 3 \cdot 1.3946 + 1.183}{8}\right] + \left[0.25 \cdot \frac{1.1831 + 4 \cdot 0.9808 + 0.8031}{6}\right] = \boxed{0.791282}$$

21.14: $\int_{-2}^{2}\int_0^4 (x^2 - 3y^2 + xy^3)\, dx\, dy$

a) $\int_{-2}^{2} \left[\frac{x^3}{3} - 3xy^2 + \frac{x^2}{2}y^3\right]_0^4 dy = \int_{-2}^{2} \frac{64}{3} - 12y^2 + 8y^3\, dy$

$\rightarrow \left[\frac{64}{3}y - 4y^3 + 2y^4\right]_{-2}^{2} = \frac{64}{3}$

b) $n = 2$ Trapezoidal rule:

at $y = 2 \rightarrow I \cong 4 \cdot \frac{-12 + 2(4 - 12 + 16) + 36}{4} = 40$

at $y = 0 \rightarrow I \cong 4 \cdot \frac{0 + 8 + 16}{4} = 24$

at $y = -2 \rightarrow I \cong 4 \cdot \frac{-12 + 2(-24) + (-28)}{4} = -88$

next integrate over y : $4 \cdot \frac{40 + 2(24) - 88}{4} = 0$

$\rightarrow \%\varepsilon_t = 100\%$

c) at $y = -2$: $4 \cdot \frac{-12 + 4(-24) - 28}{6} = -90.666\ldots7$

at $y = 0$: $4 \cdot \frac{0 + 4 \cdot 4 + 16}{6} = 21.33$

at $y = 2$: $4 \cdot \frac{-12 + 4(8) + 36}{6} = 37.3333$

integrate over y : $4 \cdot \frac{21.333 + 37.333 + 4(-90.669)}{6} = 21.3333$

$\therefore \%\varepsilon_t \approx 0\%$ ✓

21.2 : $\int_0^3 (1 - e^{-x})\,dx = \left[x + e^{-x}\right]_0^3 = 3 + e^{-3} - 1 = 2.0497871$

b) $3 \cdot \frac{0 + 0.9502129}{2} = 1.425321$

c) $3 \cdot \frac{0 + 2 \cdot 0.77687 + 0.9502129}{4} = 1.8777645$ at $n = 2$

at $n = 4$: $3 \cdot \frac{0 + 2(0.527633 + 0.776869 + 0.894601) + 0.9502129}{8}$

$= 2.0096579$

d) $3 \cdot \frac{0 + 4(0.776869) + 0.9502129}{6} = 2.028846$

e) at $n = 4$: $3 \cdot \frac{0 + 4(0.527633 + 0.894601) + 2(0.776869) + 0.9502129}{12}$

$= 2.04822376$

f) $3 \cdot \frac{0 + 3 \cdot (0.6321) + 3(0.8646) + 0.9502129}{8} = 2.040213$

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example 23.1: $h = 0.25$, $x_0 = 0.5$

$x_{i-2} = 0$, $f(x_{i-2}) = 1.2$

$x_{i-1} = 0.25$, $f(x_{i-1}) = 1.1035156$

$x_i = 0.5$, $f(x_i) = 0.925$

$x_{i+1} = 0.75$, $f(x_{i+1}) = 0.63632813 5$

$x_{i+2} = 1$, $f(x_{i+2}) = 0.2$

1) forward: $\frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2 \cdot 0.25} = -0.859375$

$\rightarrow \%\varepsilon_t = 5.8219\%$

2) backward: $\frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} = -0.8781248$

$\rightarrow \%\varepsilon_t = 3.969\%$

3) centered: $\frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} = -0.912499933$

$\rightarrow \%\varepsilon_t = 7.31 \times 10^{-6}\% \approx 0\%$

practice problems:

23.1: $x_{i-2} = \frac{\pi}{12} \rightarrow f(x_{i-2}) = 0.2588190451$

$x_{i-1} = \frac{\pi}{6} \rightarrow f(x_{i-1}) = 0.5$

$x_i = \frac{\pi}{4} \rightarrow f(x_i) = \frac{\sqrt{2}}{2}$

$x_{i+1} = \frac{\pi}{3} \rightarrow f(x_{i+1}) = \frac{\sqrt{3}}{2}$

$x_{i+2} = \frac{5}{12}\pi \rightarrow f(x_{i+2}) = 0.9659258263$

forward simple: 0.60702, forward accurate: 0.7197408

backward simple: 0.7910896314, backward accurate: 0.7260177532

$\%\varepsilon_{tFS}$: 14.15%, $\%\varepsilon_{tFA}$: 1.78672%, $\%\varepsilon_{tBS}$: 11.877%

$\%\varepsilon_{tBA}$: 2.6737%

centered simple: 0.699057 → $\%\varepsilon_t$: 1.138%

centered accurate: 0.7069969579 → $\%\varepsilon_t = 0.0153\%$

forward centered backward

23.9:	0	25	50	75	100	125
	0	32	58	78	92	100

at 0 using forward
0

velocity $\frac{-58+4\cdot 32-3\cdot 0}{50} = 1.4 m/s$

acceleration $\frac{-78+4\cdot 58-5\cdot 32+2\cdot 0}{25^2} = -9.6\times10^{-3} m/s^2$

at 25 using forward:
32

velocity: 1.16 m/s , acceleration: $-9.6\times10^{-3} m/s^2$

at 50 using centered:
58

velocity: 0.92 m/s , acceleration: $-9.6\times10^{-3} m/s^2$

at 75 using centered:
78

velocity: 0.68 m/s , acceleration: $-9.6\times10^{-3} m/s^2$

at 100 using backward:
92

velocity: 0.44 m/s , acceleration: $-9.6\times10^{-3} m/s^2$

at 125 using backward:
100

velocity: 0.2 m/s , acceleration: $-9.6\times10^{-3} m/s^2$

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- will focus on linear initial value problems
- assume $\frac{d\,y(x)}{dx} = f(x,y)$, $y(x_0) = y_0$

  Then the estimates of the solution are computed at different base points using the truncated taylor series expansion: $y(x_0+h)$, $y(x_0+2h)$

- nth order taylor series method uses the nth order truncated taylor series expansion:

$$y(x_0+h) \approx y(x_0) + h \left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} + \frac{h^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0\\y=y_0}} + \dots + \frac{h^n}{n!}\left.\frac{d^ny}{dx^n}\right|_{\substack{x=x_0\\y=y_0}}$$

$$\text{or}\quad y(x_0+h) \approx \sum_{k=0}^{n} \frac{h^k}{k!}\left(\left.\frac{d^ky}{dx^k}\right|_{\substack{x=x_0\\y=y_0}}\right)$$

- euler method uses the first order taylor series expansion, which gives the error: $O(h^2)$

$$\therefore\quad y(x_0+h) = y(x_0) + h\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} + O(h^2)$$

$$\rightarrow\quad y_{i+1} = y_i + h\,f(x_i, y_i)$$

example: $\frac{dy}{dx} = 1+x^2$, $y(1) = -4$, determine $y_{i=1\to3}$ for $h=0.01$

$\therefore x_0 = 1$, $y_0 = -4 \rightarrow y_1 = -4 + 0.01\cdot2 = -3.98$

$y_2 = y_1 + h\cdot f(x_1, y_1) = -3.98 + 0.01\cdot2.0201 = -3.959799$

$y_3 = y_2 + h\cdot f(x_2, y_2) = -3.959799 + 0.01\cdot[1+1.02^2]$
$= -3.939395$

exact: $\frac{dy}{dx} = 1+x^2 \rightarrow y = x + \frac{x^3}{3} + c$
at $x=1$ $y=-4 \rightarrow c = \frac{-16}{3}$

+ types of errors:

1- local truncation error: error from truncated taylor series in one step

2- global truncation error : accumulated over many steps

3 - round-off error :

* second order taylor series method :

$$\frac{dy(x)}{dx} = f(y,x) , \quad y(x_0) = y_0$$

$$\rightarrow \quad y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + O(h^3)$$

where $\frac{d^2y}{dx^2}$ needs to be derived analytically from $f(y,x)$

example :

$$\because \frac{dx}{dt} + 2x^2 + t = 1 \quad \rightarrow \boxed{\frac{dx}{dt} = 1-2x^2-t} \quad \rightarrow \frac{d^2x}{dt^2} = -4x\frac{dx}{dt} - 1$$

$$\rightarrow \frac{d^2x}{dt^2} = -4x\left[1-2x^2-t\right] - 1$$

$$x_{i+1} \approx x_i + h\frac{dx}{dt} + \frac{h^2}{2}\frac{d^2x}{dt^2}$$

$$x_1 \approx 1 + 0.01\left[1-2(1)^2-0\right] + \left[-4(1)\left[1-2(1)^2-0\right]-1\right]\frac{(0.01)^2}{2}$$

$$\rightarrow x_1 \approx 0.99015$$

$$x_2 \approx 0.99015 + 0.01\left[1-2(0.99015)^2-0.01\right] + \frac{(0.01)^2}{2}\left[-4(0.99015)\left[1-2(0.99015)^2-0.01\right]-1\right] = 0.98054$$

$$x_3 \approx 0.97138$$

- assume $y_{i+1} = y_i + h\phi$, for euler's method, $\phi = f(x_i, y_i)$

* midpoint method :

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(x_i, y_i)$$

$$\rightarrow \quad y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

s.t. $f(x,y) = \dot{y}(x)$ (derivative) , $y(x_0) = y_0$

- local truncation error : $O(h^3)$
- global truncation error : $O(h^2)$
- hence, the accuracy of the midpoint method is comparable to the

accuracy of the second order taylor series method.

example: $f(x,y) = 1 + x^2 + y$, $y(0) = 1$, $h = 0.1$

$\rightarrow y_{i+\frac{1}{2}} = y_i + 0.05 \cdot f(x_i, y_i)$

$\rightarrow y_{0.5} = 1 + 0.05 \cdot [1 + 0^2 + 1] = 1.1$

$\therefore y_1 = y_0 + 0.1 \cdot [1 + (0.05)^2 + 1.1] = 1.21025$

$\rightarrow y_{1.5} = 1.21025 + 0.05 \cdot [1 + 0.1^2 + 1.21025] = 1.3212625$

$\therefore y_2 = 1.21025 + 0.1 \cdot [1 + (0.15)^2 + 1.3212625] = 1.44462675$

✱ heun's predictor corrector method:

- a prediction is first made, then the average slope of the initial point's and the predicted point's slopes is obtained and used to calculate our second point:

$y'(x) = f(x, y)$, $y(x_0) = y_0$

predictor: $y^0_{i+1} = y_i + h f(x_i, y_i)$

corrector: $y^1_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})\right]$

example: $y'(x) = 1 + x^2 + y$, $y(0) = 1$, $h = 0.1$

$\rightarrow y^0_{i+1} = 1 + 0.1(2) = 1.2$

$\therefore y^1_1 = 1 + 0.05\left[(1 + 0^2 + 1) + (1 + (0.1)^2 + 1.2)\right]$

$= 1.2105$

$\rightarrow y^0_2 = 1.2105 + 0.1[1 + 0.1^2 + 1.2105] = 1.43255$

$\therefore y^1_2 = 1.2105 + 0.05[2.2205 + (1 + (0.2)^2 + 1.43255)]$

$= 1.4451925$

Q1) $f(x) = 0.5x + 3\sqrt{x}$, $x_0 = 1.08$

a) $f(x_1) = f(x_0) + f'(x_0)\cdot h + \frac{f''(x_0)}{2}\cdot h^2$

$h = 0.5$, $f'(x_0) = 0.5 + \frac{3}{2\sqrt{x_0}}$

$f''(x_0) = \frac{-3}{4x_0^{3/2}}$

$\rightarrow f(x_1) = 3.65769 + 0.9716898 - 0.083529$

$\rightarrow f(x_1)_{app} = 4.5458488$

b) ∴ $R = f_{exact} - f_{app}$, $f_{exact} = 4.56094$

$\rightarrow R = 0.015093$

Q2) b) $\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 0.5$ | $y_0 = 1$

$x_0 = 4.25$

$\frac{\partial v}{\partial x} = -3e^{-x}$ $\frac{\partial v}{\partial y} = 1$

$u_0 = 13.5625$ $v_0 = -0.95721$

$$\rightarrow \begin{bmatrix} 8.5 & 0.5 \\ -0.042743 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -13.5625 \\ 0.95721 \end{bmatrix}$$

$\rightarrow x_1 = 2.60225$, $y_1 = 1.8866979$

$\rightarrow u_1 = 2.715054$, $v_1 = 0.085302$

$$\therefore \begin{bmatrix} 5.430108 & 0.5 \\ -0.198604 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -2.715054 \\ -0.085302 \end{bmatrix}$$

$\rightarrow x_2 = 2.118943$

Q3) a) ∴ $3a_0 + \sum^n x_i \cdot a_1 = \sum y_i$

$\rightarrow 1.5 + 4.5\cdot 2 = 6.5 + y_3$

$\rightarrow y_3 = 4$

Or $\bar{x} = 1.5$ $a_0 = \frac{2.2 + 4.3 + y_3}{3} + a_1\cdot\bar{x}$

$\rightarrow 0.5 + 3 = \frac{2.2 + 4.3 + y_3}{3}$

$\rightarrow y_3 = 4$

$S_r = 1.15 = \sum_{i=1}^{3} (y_i - a_0 - a_1 x_i)^2 = 0.09 + 0.64 + (y_3 - 4.5)^2$

$\rightarrow y_3 - 4.5 = \sqrt{0.42} \rightarrow y_3 = 5.1481$

b) $S_{y/x} = \sqrt{\frac{1.15}{1}} = 1.07238$

c) $S_t = (y - \bar{y})^2 \quad \bar{y} = 3.8827 \rightarrow S_t = 4.60685574$

d) $r = 0.750372$

Q3) a) $S_r = 2.38 = 0.73 + (y_3 - 4.5)^2 \rightarrow y_3 = 5.784523$

b) $S_{y/x} = 1.543 \qquad \bar{y} = 4.09484$

c) $S_t = 6.489537$

d) $r = 0.633143$

Q4) $t_{ii} = 2.49$

$$\begin{bmatrix} n & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i t_i \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 2.49 \\ 2.49 & 6.4901 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 8.35 \\ 11.383 \end{bmatrix}$$

$\rightarrow a_0 = 1.904206 \ , \ a_1 = 0.882067$

① $a_0 \cdot \log(0) + a_1 \cdot e^{0} = 2.3$

② $a_0 \cdot \log(0.5) + a_1 e^{0.05} = 1.85$

③ $a_0 \log(2.41) + a_1 \cdot e^{0.241} = 4.2$

$$\rightarrow [Z] = \begin{bmatrix} 1 & 1 \\ 0.8994825 & 1.0512711 \\ -0.7951189 & 1.2827942 \end{bmatrix}$$

$\rightarrow \{A\} \Rightarrow a_0 = -0.79362095 \qquad a_1 = 2.95922071$

Q5) $n = 4 \ , \ I = \frac{f(x_0) + 2\sum_{i=1}^{3} f(x_i) + f(x_4)}{} \ , \ B = 3.8$

$\rightarrow I = \frac{6.8 + 2[7.87929 + 9.265181 + 11.0446] + 13.32471}{8}$

$\rightarrow I = 9.5634416$

Q5) b) $\frac{6.8 + 4 \cdot 9.265141 + 13.329471}{6} = 9.5316725$

Q6) b) $\frac{dy}{dx} = ax + y^2$ $a = 3.5$, $h = 0.1$, $y_0 = 2$, $x_0 = 1$

$\rightarrow y_1 = 2 + 0.1 \cdot (3.5 + 4) = 2.75$

$\rightarrow y_2 = 2.75 + 0.1 \cdot (3.5 \cdot 1.1 + (2.75)^2) = 3.89125 = y(1.2)$

c) $y_{0.5} = 2.375$ $\rightarrow y_1 = 2 + 0.1(3.5 \cdot 1.05 + 2.375^2)$

$= 2.9315625$

$y_{1.5} = 2.9315625 + 0.05(3.5 \cdot 1.1 + 2.9315625^2)$

$= 3.553765$

$\rightarrow y_2 = 2.9315625 + 0.1(3.5 \cdot 1.15 + 3.553765^2)$

$= 4.54698907$

d) $y_1^0 = 2.75$

$y_1^1 = 2 + 0.05[7.5 + 11.4125] = 2.945625$

$y_2^0 = 4.198295664$

$\rightarrow y_2^1 = 2.945625 + 0.05[12.5269 + 21.8256865]$

$\rightarrow y_2^1 = y(1.2) = 4.663244325$

Q1) a) $\frac{dy}{dx} = y^2 + 0.4e^{-x}$, $x_0 = 2$, $y_0 = 4.7$, $h = 0.3$

$\rightarrow y_1 = 4.7 + 0.3[4.7^2 + 0.4e^{-2}] = 11.34324$

b) $y_{0.5} = 4.7 + \frac{0.3}{2}[22.14413411] = 8.021620117$

$\rightarrow y_1 = 4.7 + 0.3[8.0216^2 + 0.4e^{-2.15}]$

$= 24.019899$

c) $y_1^0 = 11.34324$

$\rightarrow y_1^1 = 4.7 + 0.15[22.14413411 + 128.90414]$

$\rightarrow y_1^1 = 27.327948$

diverging ?

Q2) a) forward: $f'(2) = \frac{-f(2.4)+4f(2.2)-3f(2)}{0.2\cdot 2}$

$= \frac{-2.4+8.4-6.3}{0.4} = -0.75$

b) centered: $\cancel{\frac{-2.5+16\cdot 2.7-30\cdot 2.2+16\cdot 2.4-2.1}{12}}$ not equally spaced

forward: use normal centered

$\rightarrow f''(2.8) = \frac{2.7-2\cdot 2.2+2.4}{0.4^2} = 4.375$

c) use accurate backward:

$f''(3.6) = \frac{2\cdot 2.5-5\cdot 2.7+4\cdot 2.2-2.4}{0.4^2}$

$= -13.125$

Q3) $\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 4y \quad \frac{\partial v}{\partial x} = -10e^{-x}+y$

$\frac{\partial v}{\partial y} = x \quad , \quad x^0 = 2.25 \quad , \quad y^0 = 1$

$u_0 = -0.75 \wedge v_0 = 1.30399225$

$$\begin{bmatrix} 1 & 4 \\ -0.0539922 & 2.25 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.3039922 \end{bmatrix}$$

a, b) $\rightarrow x_1 = 5.049495 \wedge y_1 = 0.4876262$

$$\begin{bmatrix} 1 & 1.950505 \\ 0.4235005 & 5.049495 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.525054 \\ -0.52639197 \end{bmatrix}$$

c, d) $\rightarrow x_2 = 4.664851 \wedge y_2 = 0.41564$

e) $\left|\frac{4.664851-5.049495}{4.664851}\right| \times 100 = 8.2455\%$

Q4) $x = 0.21$, $y = 0.3$, $z = 0.5$

a) $(xz)/y$

$x \to 0.17$, $y \to 0.25$, $z \to 0.4$

$xz \to 0.05$ $\frac{xz}{y} = 0.17$

b) $\frac{xz}{y}$ true $= 0.35$

$\to \% \varepsilon_t = 51.4286\%$

Q5)

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 4 & 5 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

Jacobi : use initial set then update all the set

a) $x_1^1 = \frac{2 - 1 \cdot 1 - 2}{4} = -0.25$ $x_2^1 = \frac{7 - 2 + 10}{9} = \frac{5}{3}$

b) $x_3^1 = \frac{4 - 1 - 3}{6} = 0$

c) $x_1^2 = \frac{2 - \frac{5}{3}}{4} = \frac{1}{12}$

d) $x_3^2 = \frac{4 + 0.25 - 5}{6} = \frac{-1}{8}$

e) 400 %

first 2020:

$\boxed{1}$ a) $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $x_i = 0.15$

$\rightarrow x_{i+1} = 0.15 - \frac{4.778375}{-2.9325} = 1.7796544$

b) $\%RE_t = -142.23$

c) $x_{i+2} = x_{i+1} - \frac{f(x_{i+1})}{f'(x_{i+1})} = 1.7794544 - \frac{-21.03}{-26.0997}$

$= 0.97339$

d) $\%RE_a = \frac{0.97339 - 1.7794544}{0.97339} = -82.81\%$

$\boxed{2}$ $f(x) = \frac{1}{x-1} \rightarrow f'(x) = \frac{-1}{(x-1)^2}$, $f''(x) = \frac{2}{(x-1)^3}$

a) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$

$f(x_i) = \frac{10}{3}$, $f'(x_i) = -\frac{100}{9}$, $f''(x_i) = \frac{1000}{27}$

$\rightarrow f(x_{i+1}) = \frac{10}{3} + R_2$

b) $\because \overset{\text{exact}}{f(x_{i+1})} = \frac{5}{3} \rightarrow R_2 = \frac{-5}{3}$

c) $\left| \frac{\overset{\text{true}}{f(x_{i+1})} - \overset{\text{app}}{f(x_{i+1})}}{f(x_{i+1})} \right| = \cancel{100}\ 100$

d) $\because R_2 = \frac{f'''(\xi)}{(n+1)!} \cdot h^3$, $f''' = \frac{-6}{(x-1)^4}$

$\rightarrow R_2 = \frac{-6/(\xi-1)^4}{6} \cdot (0.3)^3 = \frac{-5}{3}$

$\rightarrow \frac{-3}{5} = \frac{-(\xi-1)^4}{(0.3)^3} \Rightarrow \xi = 1.3568$

$\boxed{3}$ $f(x) = \ln(x) \rightarrow f'(x) = \frac{1}{x} = \cancel{1.4}$

a) $x_0 = 0.9 \rightarrow f'(x_0) = \frac{10}{9}$

b) $f'_{app} = \frac{\ln(0.9) - \ln(0.5)}{} = 1.6820$

c) $f'_{app} = \frac{-0.35 - -0.69 \cdot 2}{0.2} = \frac{0.34}{0.2} = 1.7$

d) $\cancel{1.7} - \left(\frac{10}{9} - 1.9\right) = 0\ 291478$

④ a) $x = -0.090081$

b) $\frac{-20 + \sqrt{400 - 1.44}}{0.4} = \frac{-20 + 19.96}{0.4} = -0.1$

c) 11.01%

2018:

$\boxed{1}$ $f'(x)^{true} = 6.0294118476$

$f'(x_i)^{approx} = \frac{f(2) - f(1.5)}{0.5} = \frac{7.915 - 4.349}{0.5} = 7.136$

$\to \%RE_t = \frac{6.029 - 7.136}{6.029} = 18.36\%$

$\boxed{2}$ taylor series:

$f(1.1) = f(1) + f'(1)\cdot 0.1 + \frac{f''(1)}{2}\cdot(0.1)^2 + \frac{f^{(3)}(1)}{6}\cdot(0.1)^3$

$\to f(1.1) = -2 + 0.4 + 0 - \frac{1}{2000} + \frac{5}{24}\cdot(0.1)^4$

$= -1.600\cancel{47} = -1.600$

$\boxed{3}$ $f(x) = 8\sin(x)\, e^{-x} - 1$, $f'(x) = -8e^{-x}(\sin(x) - \cos(x))$

$\to f(x) = 8\sin(x)\cdot 2.7182^{-x} - 1$

$f'(x) = -8\cdot 2.7182^{-x}[\sin(x) - \cos(x)]$

1) $f(x_0) = 8\cdot 0.2955\cdot 0.7408 - 1 = 0.7749$

$f'(x_0) = -8\cdot 0.7408\,[0.2955 - 0.9553] =$

$\to 0.3 - \frac{f(x_0)}{f'(x_0)} = 0.3 - \frac{0.7749}{3.9096} = 0.1017$

2) $f(x_1) = 8\cdot 0.1015\cdot 0.9033 - 1 = -0.2666$

$f'(x_1) = 8\cdot 0.9033\,[0.1015 - 0.9948] = -6.4552$

$\to$ ~~f~~ $x_2 = 0.0604$

Q) $x_1 = 0.3 - \frac{0.75141}{3.9103} = 0.10784$

$x_2 =$

$\frac{-0.22699}{6.3674}$

0.75141

not chopping: 0.75139

3.9085

4 $x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i+\delta x_i) - f(x_i)} = \frac{2.2541\times10^{-3}}{~~2.2842\times10^{-3}~~ - ~~[illegible]\times10^{-3}~~}$

$0.96309 - 0.75161$

$\rightarrow x_1 = 0.10700$

$\rightarrow x_2 = x_1 - \frac{-2.4859\times10^{-6}}{-0.22548 - -0.23230} = 0.163065$

5 $x_{n1} = 5.25$

$f x_n \cdot f x_1 = -3.6875 \cdots \rightarrow >0 \rightarrow x_i = x_{n1}$

$\rightarrow x_{n2} = 5.625$

6 $x_n = 6 - \frac{7\cdot -1.5}{-3.6875 - 7} = 5.0175$

$f(x_n)\cdot f(x_l) > 0 \rightarrow x_l^{new} = x_n$

$\rightarrow x_n^2 = 6 - \frac{7\cdot 0.98245}{-1.001 - 7} = 5.1604$

7 $2^{+7} \cdot -(111) = -2^7 \cdot 0.875 = \boxed{-112}$

8 0.1639×10^{-1}

9 $2^{-15} \cdot (-1001) = -1.52587.06\times10^{-5}$

10 $2^{-15} \cdot (10000) =$

$8\, \sin(x)\, e^{-x} - 1$

0.10784: $8 \cdot 0.10763 \cdot 0.89779 - 1 = -0.22698$

0.10785: $8 \cdot 0.10764 \cdot 0.89776 - 1 = -0.22692$

0.89776 changed e

0.10784: $-7.1821 \cdot -0.88656 = 6.3673$

0.10785: $-7.1120~~8~~ \cdot -0.88654 = 6.3671$

0.74082

−5.9266

−0.66951

0.10763 − 0.99419

0.10764 − 0.99418

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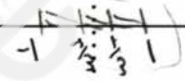
MOHAMMAD SANAD ALTAHER 130806

Q1)

a) $\int_{-1}^{1} \left[0.7x^2 + e^{-x}\right] dx = \boxed{2.817064}$

b) $I \simeq (b-a)\dfrac{f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)}{6}$

$x_1 = -\frac{1}{3}$, $x_2 = \frac{1}{3}$



$\rightarrow I = \frac{2}{6}\left[3.41828 + 2(1.47339) + 2(0.7443)\right.$
$\left. + 1.067879\right]$

$= \boxed{3.0072}$

c) $E_t = 2.817064 - 3.00728 = \boxed{-0.19011}$

d) $I \simeq (b-a)\left[\dfrac{f(-1) + 4f(0) + f(1)}{6}\right]$

$\rightarrow I = \boxed{2.8287204}$

e) $E_t = 2.817064 - 2.8287204 = \boxed{-0.01165}$

N O T E B O O K

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Q2) exponential → $y = a e^{Bx}$

→ $\ln(y) = \ln(a) + Bx$, $n=3$

$\sum \ln(y) = 2.90243446$, $\sum x = 5.8$

$\sum x^2 = 10.8$ $\sum \ln(y) x = 4.9953195$

$$\begin{bmatrix} 3 & 5.8 \\ 5.8 & 10.8 \end{bmatrix} \begin{bmatrix} \ln a \\ B \end{bmatrix} = \begin{bmatrix} 2.90243446 \\ 4.9953195 \end{bmatrix}$$

→ $\boxed{B = 0.194123}$, $\boxed{a = 1.8202934}$

c) $S_r = \sum (y_{measured} - y_{model})^2$

$= (2.26 - 2.166)^2 + (2.2 - 2.3227)^2$
$+ (3 - 2.9640)^2 = 0.02511246$

floating point = $\boxed{0.2511 \times 10^1}$

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Q3) assuming $h = +0.4$, $x_0 = 2$, $y_0 = 2.8$

a) $y_1 = y_0 + h\left[y_0^2 + 0.4e^{-x_0}\right]$

$\rightarrow y_1 = 5.9576536$

b) $\because y_0^2 + 0.4e^{-x_0} = 7.894134$

$\rightarrow y_{0.5} = 2.8 + 0.2\left[7.894134\right] = 4.378826$

$\therefore y_1 = 2.8 + 0.4\left[4.378826^2 + e^{-2.2}\right]$

$= 10.489378$

c) predictor $\Rightarrow y_1^0 = 5.9576536$

corrector $\rightarrow y_1^1 = 2.8 + 0.2\left[7.894134 + 5.9576536^2 + 0.4e^{-2.4}\right]$

$\rightarrow \boxed{y_1^1 = 11.48481}$

MOHAMMAD SANAD ALTAHER 130806

Q4) $u(x, y) = x + y^2 - 5$ $(1.94, 1)$

$v(x, y) = 10e^{-x} + xy - 2$

$\rightarrow \frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = -10e^{-x} + y$

$\frac{\partial v}{\partial y} = x$

a)b) $u_0 = -2.26$, $v_0 = 1.495204$

$$\rightarrow \begin{bmatrix} 1 & 2 \\ -0.9552 & 1.94 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 2.26 \\ -1.495204 \end{bmatrix}$$

$\rightarrow x^1 = 3.869827$, $y^1 = 1.0650863$

c)d) $u_1 = 4.2358 \times 10^{-3}$, $v_1 = 2.3303145$

$$\rightarrow \begin{bmatrix} 1 & 2.1301726 \\ 0.8564665 & 3.869827 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -4.2358 \times 10^{-3} \\ -2.3303145 \end{bmatrix}$$

$\therefore x^2 = 6.288629$, $y^2 = -0.092436$

e) $$\left| \frac{6.288629 - 3.869827}{6.288629} \right| \times 100 = 38.4631\%$$